Synchronization: What can a plasma physicist say about generic collective behavior?

Thomas M. Antonsen Jr.
and
Edward Ott

Happy Birthday Nat!
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Generic behavior involving a large ensemble of nearly identical oscillators that are weakly coupled.

- Cellular clocks in the brain.
- Pacemaker cells in the heart.
- Flashing fire flies.
- Deep Brain Stimulation (DBS) treatment for Parkinson’s.
- Pedestrians on a bridge
- Many more
Activity of cells in the Suprachiasmatic Nucleus (SCN)

The daily pacemaker

Dorsal cells don’t synchronize
Top trace shows activity with multiple frequencies and phases

Ventral cells synchronize
Lower trace shows locked frequencies, but various phases

Cellular clocks in the brain (day-night cycle).
Yamaguchi et al., Science 302, 1408 (‘03).
Phase Oscillators

Dynamics of individual elements is in higher dimensional space

Attraction to limit cycle
Period $T_i = 2\pi/\omega_i$

With weak coupling to others, only the evolution of the phase is important.

$$\frac{d\theta_i(t)}{dt} = \omega_i + \text{coupling to others}$$

Competition between effects of coupling and spread in distribution of natural frequencies - Winfree
Kuramoto Model (1975)

\[ \frac{d\theta_i(t)}{dt} = \omega_i + \frac{k}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \]

Individual frequencies selected from a pdf \( g(\omega) \).

\( k = \) coupling strength  \hspace{1cm} \text{Phases attract each other}

Introduce order parameter

A “mean” field

\[ R(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j) \]

\[ \frac{d\theta_i(t)}{dt} = \omega_i + k \text{Im}\left(\text{Re}^{-i\theta_i}\right) \]
Order parameter measures the coherence

\[ R(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j) \]

\[ |R| \approx 1 \quad |R| \approx 0 \]
Spontaneous Synchronization

There is a transition to synchrony at a critical value of the coupling constant.

Order parameter decays in time

Incoherence

Order parameter grows in time and saturates.

Synchronization
\[ N \rightarrow \infty \text{ Limit: Shades of Vlasov} \]

**Distribution Function**

\[ f(\theta, \omega, t) \quad g(\omega) = \int_{0}^{2\pi} f(\omega, \theta, t) d\theta \]

**Evolution of DF**

\[ \frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta} \left\{ \frac{d\theta}{dt} f \right\} = 0 \]

**Flow in \( \theta \)**

\[ \frac{d\theta}{dt} = \omega + K \text{Im}(\text{Re}^{-i\theta}) \]

**Flow in \( \theta \) (Continued)**

\[ R(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j) \rightarrow \int_{0}^{2\pi} \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\omega f e^{i\theta} \]

*Where did the \( d\omega/dt \) go?*
Connection to Landau Damping

Early attempts to understand onset of synchronization led to confusion
“Incoherence was neutral by one measure stable by another”. S.S

Linear theory:
\[ R(t) = \int \int d\omega d\theta f(\omega, \theta, t = 0)e^{-i\omega t} + Ae^{st} \]
As \( t \to \infty \)
Ballistic term
Collective mode

Collective Mode
Dispersion Relation
\[ 1 = \frac{k}{2} \chi(s) = \frac{k}{2} \int d\omega \frac{g(\omega)}{s - i\omega} \]

S. Strogatz, Sync, p64.
“Paul (Matthews) gave a lecture at the University of Warwick... One of the Professors in the audience, George Rowlands, told Paul that what we were seeing was not so strange: It’s called Landau damping, and plasma physicists have known about it for 45 years.”
Spontaneous Synchronization

There is a transition to synchrony at a critical value of the coupling constant.

\[ g(\omega) = \text{PDF of natural oscillator frequencies } \omega. \]

\[ 1 = \frac{k}{2} \int d\omega \frac{g(\omega)}{s - i\omega} \quad s \to i\omega_0 + 0 \]

\[ = \frac{k}{2} \pi g(\omega_0) \]

\[ k_c = \frac{2}{\pi g_{\max}} \]
What We Found

A nonlinear solution for the distribution function in the limit $N \to \infty$.

The time evolution of the order parameter is that of a low dimensional system.

The nonlinear solution is a "weak" attractor.

Special case of Lorenzian

$$g(\omega) = \frac{\Delta}{\pi \left[ (\omega - \omega_0)^2 + \Delta^2 \right]}$$

$$\frac{dR}{dt} + \frac{k}{2} (|R|^2 - 1)R + (-i\omega_0 + \Delta)R = 0$$
Nonlinear Solution: the O-A Ansatz

A distribution function of the following form satisfies the kinetic equation.

\[ f_{OA}(\theta, \omega, t) = \frac{g(\omega)}{2\pi} \left[ \frac{1}{1 - \alpha(\omega,t)e^{i\theta}} - \frac{1}{2} \right] + c.c \]

Where:

\[ \frac{\partial}{\partial t} \alpha + i\omega \alpha + \frac{k}{2} \left( R \alpha^2 - R^* \right) = 0 \]

parametric dependence on \( \omega \)

Order Parameter:

\[ R(t) = \int d\omega g(\omega) \alpha^*(\omega, t) \]

\[ \frac{d}{dt} R(t) = \int d\omega g(\omega) \frac{\partial}{\partial t} \alpha^*(\omega, t) = \ldots \]
Weak Attractor

In general, the distribution function is the sum of the O-A Ansatz and a remainder. O-A ansatz lies on a submanifold $M$ in space of functions.

$$f(\theta, \omega, t) = f_{OA}(\theta, \omega, t) + f_{rem}(\theta, \omega, t)$$

You cannot show $f_{rem}(\theta, \omega, t) \to 0$ as $t \to \infty$

You can show

$$\int \omega d\theta \int f_{rem}(\theta, \omega, t) e^{-i\theta} \to 0 \quad \text{as} \quad t \to \infty$$

Solution for $R(t)$ is an attractor

- $M$ is an invariant submanifold.
Generalizations of the Kuramoto Model

**External Drive:**

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + M_0 \sin(\Omega_0 t - \theta_i)
\]

E.g., circadian rhythm.

Ref.: Sakaguchi, ProgTheorPhys(‘88); Antonsen, Faghih, Girvan, & Ott, Platig, Chaos 18 (‘08); Childs, Strogatz, Chaos 18 (‘08).

**Communities of Oscillators:**

\[A = \# \text{ of communities}; \quad \sigma = \text{community } (\sigma = 1, 2, \ldots, s);\]
\[N_\sigma = \# \text{ of individuals in community } \sigma.\]

\[
\frac{d\theta_i^\sigma}{dt} = \omega_i^\sigma + \sum_{\sigma' = 1}^{s} \left( \frac{k_{\sigma\sigma'}}{N_{\sigma'}} \right) \sum_{j=1}^{N_{\sigma'}} \sin(\theta_j^{\sigma'} - \theta_i^\sigma + \beta^{\sigma\sigma'})
\]

Generalizations of the Kuramoto Model

Millennium Bridge Problem:

\[
\begin{align*}
\frac{d\theta_i}{dt} &= \omega_i - b \left( \frac{d^2 y}{dt^2} \right) \cos(\theta_i + \beta) \quad \text{(Walker phase)} \\
\left( \frac{d^2 y}{dt^2} \right) + \nu \frac{dy}{dt} + \Omega^2 y &= \frac{1}{M} \sum_i f_i \\
\text{(Bridge mode)}
\end{align*}
\]

\[
\begin{align*}
f_i(t) &= f_{i0} \cos(\theta_i(t)) \\
\text{(Walker force on bridge)}
\end{align*}
\]

Ref.: Eckhardt, Ott, Strogatz, Abrams, & McRobie, PhysRevE 75, 021110 ('07); Abdulrehem and Ott, Chaos 19, 013129 ('09).
Extensions: heterogeneous time delays


\[ R(t) = \int_0^\infty d\tau \ h(\tau) \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega \ f(t - \tau) \ e^{i\theta} \]

Critical Coupling versus Mean Delay

Hysteresis with heterogeneous delays
Local spatial coupling


\[ R(t, x) = \int d^n x' q(x - x') \int_0^\infty d\tau \ h(\tau) \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega \ f(t - \tau, x') e^{i\theta} \]
Unsolved Problem: Large but finite N

TMA, E. Ott, Paul So, and Ernie Barreto

Plasma approach: dressed test particles


Applies to the incoherent state  $k < k_c$

What’s the problem?

- So far no settled theory applies to the transition  
  (Hong et al. PRE 2015, Daido PRE 2015 – scaling theory)
- To complicate matters: Time averaging and ensemble averaging are not the same!
- What is important? Time average of a single realization or ensemble average?

* Nat’s academic brother
Fluctuations at the Transition

\[ \frac{k}{k_c} \]

\[ N = \infty \]

Realization to realization variation in threshold

Dressed test particles OK
The tale of three different realizations: Manny, Mo, and Jack

The following are results based on three simulations, each with populations of 100,000 oscillators. Each simulation has its own independent, randomly sampled set of natural frequencies ($\omega$s), drawn from a Lorentzian of HWHH 1. Each also uses its own independent set of randomly chosen initial phases ($\theta$s).

I call the three cases Manny, Moe, and Jack.

Below I show results for $K=1.9$, 2.0, and 2.08.

**PLEASE NOTE THAT THE GRAPHS BELOW THAT SAY $K=2.8$ ARE INCORRECT – THEY SHOULD SAY $K=2.08$**

Temporal variations are only part of the story.
Changes is fluctuation levels

As $k$ is increased past $k_c$ distribution of $R^2$ values evolves

Different realizations transition at different $k$-values.
Plasma Physics has had, and will have a big influence on the understanding and mathematical description of synchronization.

The Kuramoto model has an exact nonlinear solution in the $N$- infinite limit that is a weak attractor.

The correct description of the finite $N$ with noise limit is still unresolved.

Oh yes, and Happy Birthday Nat.
Generalizations (continued)

- **Time varying link coupling strengths:** P. So, B. Cotton, & E. Barreto, *Chaos* 18 (‘08).


- **Josephson junction circuits:** S. Marvel, S. Strogatz, *Chaos* (‘09).

- **Oscillators distributed in space with local coupling:** This situation displays hysteresis, traveling fronts, spiral waves, target patterns, stationary spots, chimera, etc. C. Laing, *Chaos* 18 (‘09); and W. S. Lee, J. G. Restrepo, E. Ott, & T. M. Antonsen, *Chaos* 21, 023122 (2011).


- **And others.**