Happy Birthday Nat !

Synchronization: What can a plasma physicist say about generic collective behavior?

Thomas M. Antonsen Jr.

and

Edward Ott

University of Maryland 1856

Institute for Research in Electronics & Applied Physics

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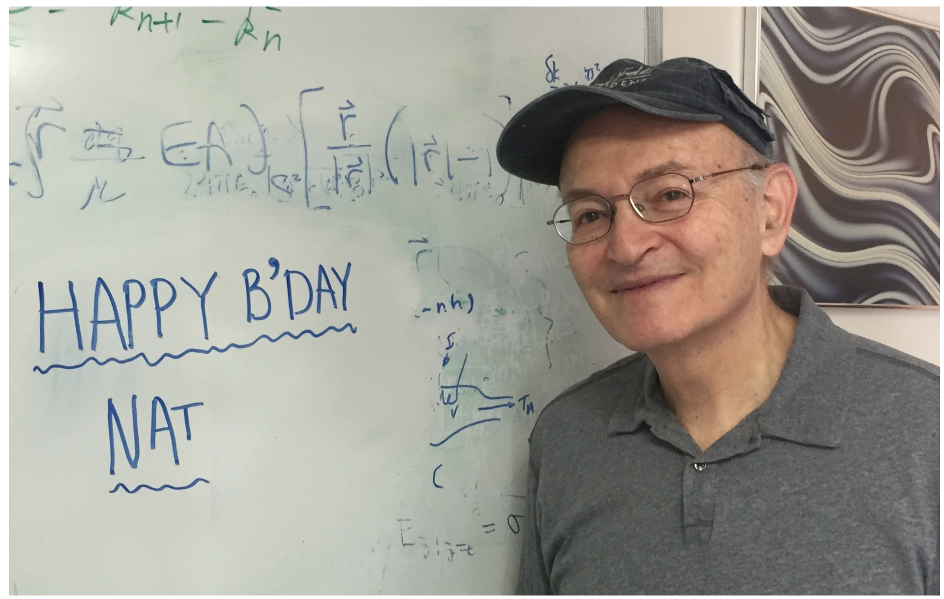

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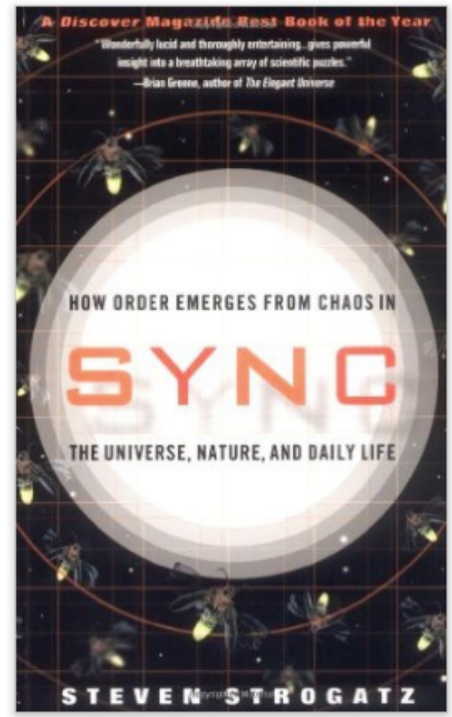
Synchronization in Nature

Generic behavior involving a large ensemble of nearly identical oscillators that are weakly coupled.

- **Cellular clocks in the brain.**
- **Pacemaker cells in the heart.**
- **Flashing fire flies.**
- **Deep Brain Stimulation (DBS) treatment for Parkinson's.**
- **Pedestrians on a bridge**
- **Many more**

by Steven Strogatz



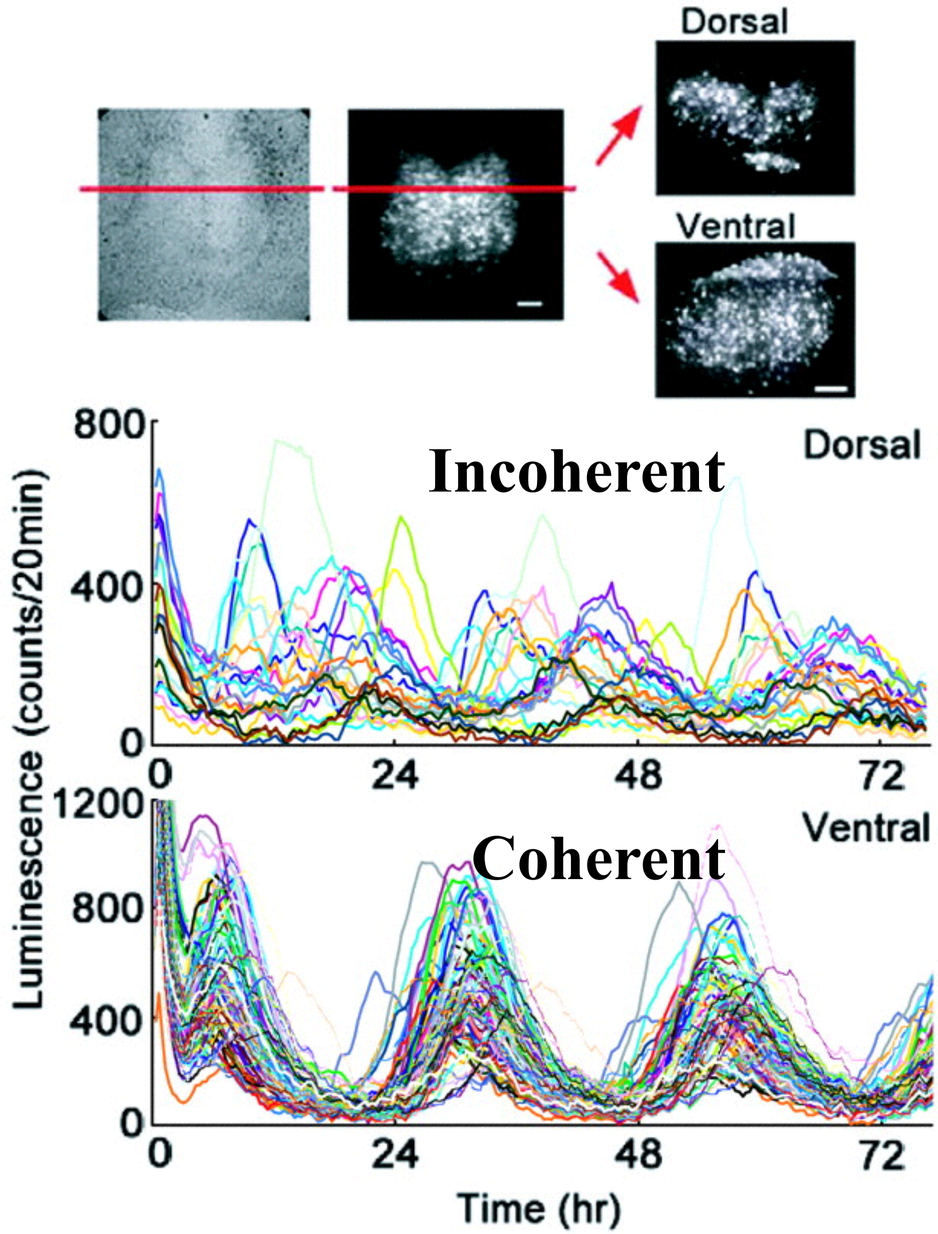
ELSEVIER

Physica D: Nonlinear Phenomena

Volume 143, Issues 1–4, 1 September 2000, Pages 1–20

From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators

Steven H. Strogatz



Cellular clocks in the brain (day-night cycle).
Yamaguchi et al., Science <u>302</u>, 1408 (‘03).

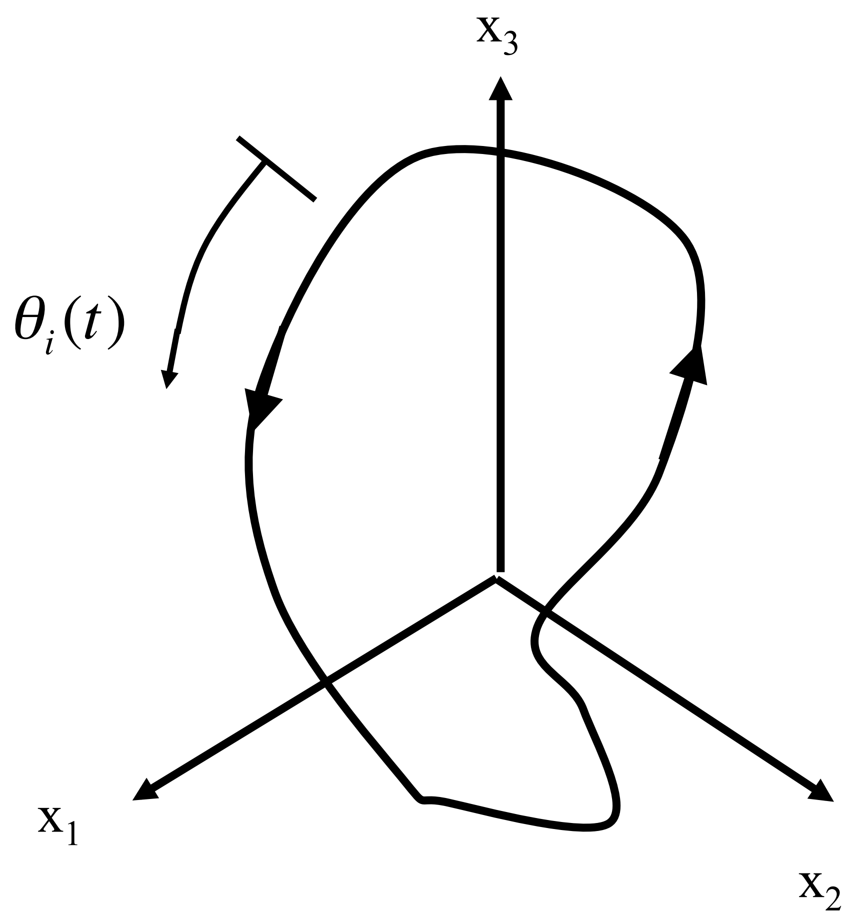
Activity of cells in the Supra-chiasmatic Nucleus (SCN)

The daily pacemaker

<u>Dorsal cells don’t synchronize</u>
Top trace shows activity with multiple frequencies and phases

<u>Ventral cells synchronize</u>
Lower trace shows locked frequencies, but various phases

Phase Oscillators



Dynamics of individual elements is in higher dimensional space

Attraction to limit cycle
Period $T_i = 2\pi/\omega_i$

With weak coupling to others, only the evolution of the phase is important.

$$\frac{d\theta_i(t)}{dt} = \omega_i + \text{coupling to others}$$

Competition between effects of coupling and spread in distribution of natural frequencies - <u>Winfree</u>

Kuramoto Model (1975)

$$\frac{d\theta_i(t)}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin\left(\theta_j - \theta_i\right)$$

Individual frequencies selected from a pdf g(ω).

k = coupling strength

Phases attract each other

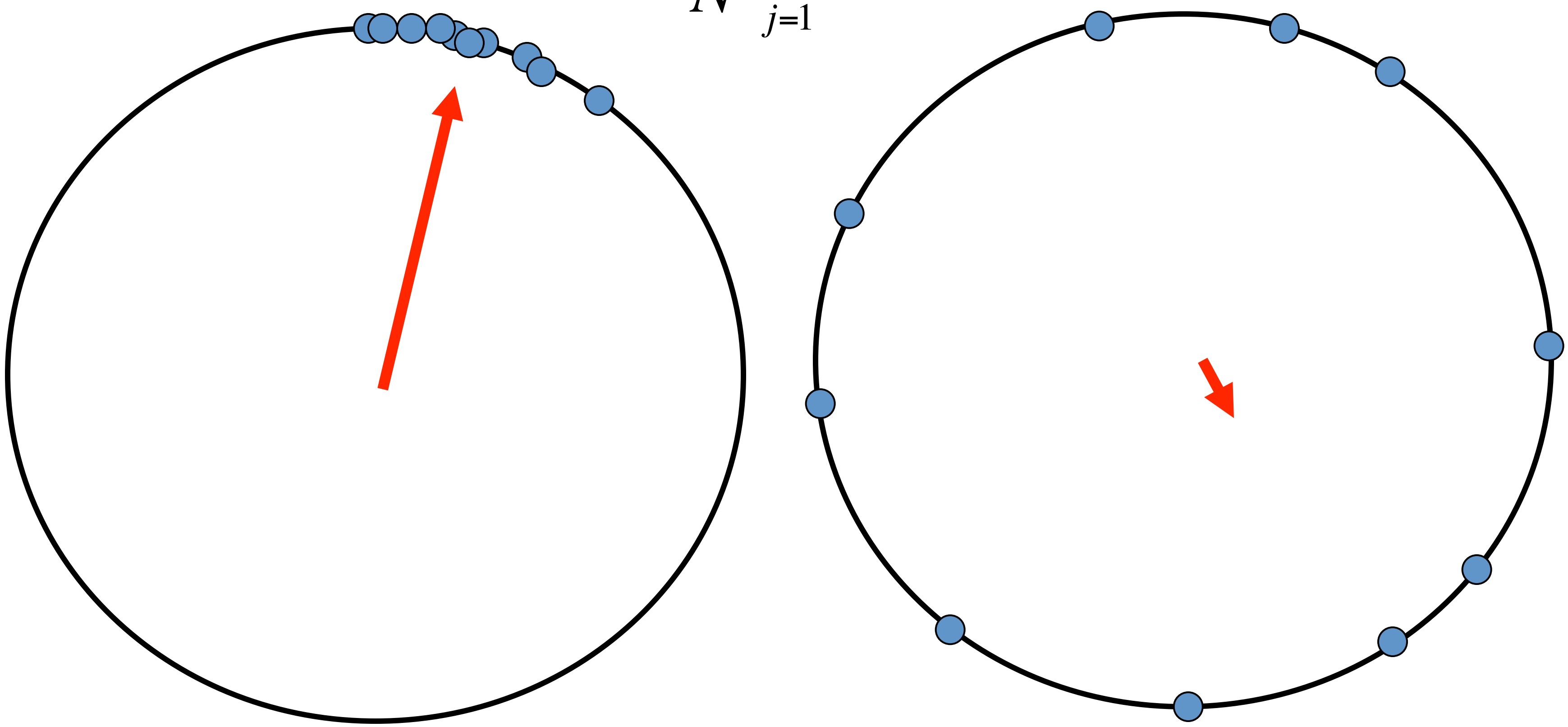
Introduce order parameter
A "mean" field

$$R(t) = \frac{1}{N}\sum_{j=1}^{N}\exp\left(i\theta_j\right)$$

$$\frac{d\theta_i(t)}{dt} = \omega_i + k\,\mathrm{Im}\left(\mathrm{R}e^{-i\theta_i}\right)$$

Order parameter measures the coherence

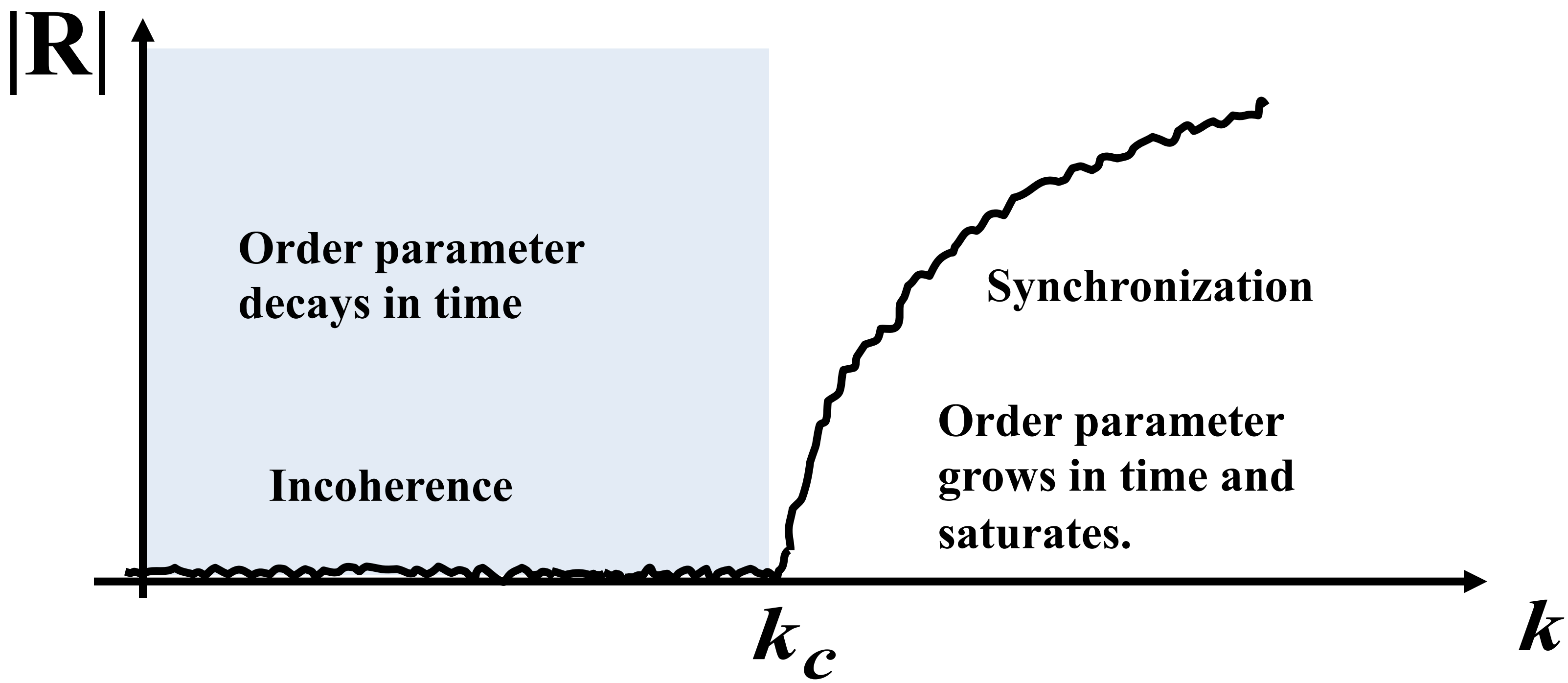
$$R(t) = \frac{1}{N}\sum_{j=1}^{N} \exp\left(i\theta_j\right)$$



$$|R| \approx 1 \qquad\qquad |R| \approx 0$$

Spontaneous Synchronization

There is a transition to synchrony at a critical value of the coupling constant.



N➔∞ Limit: Shades of Vlasov

Distribution Function $f(\theta,\omega,t)$ $g(\omega)=\int_0^{2\pi} f(\omega,\theta,t)d\theta$

Evolution of DF $$\frac{\partial f}{\partial t}+\frac{\partial}{\partial\theta}\left\{\frac{d\theta}{dt}f\right\}=0$$

Where did the dω/dt go?

Flow in θ $$\frac{d\theta}{dt}=\omega+K\,\mathrm{Im}\left(\mathrm{Re}^{-i\theta}\right)$$

$$R(t)=\frac{1}{N}\sum_{j=1}^{N}\exp\left(i\theta_j\right)\rightarrow\int_0^{2\pi}d\theta\int_{-\infty}^{+\infty}d\omega\, f\, e^{i\theta}$$

Connection to Landau Damping

Early attempts to understand onset of synchronization led to confusion
"Incoherence was neutral by one measure stable by another". .S.S

Linear theory:
As t➔ ∞

$$R(t) = \iint d\omega\, d\theta\, f(\omega,\theta,t=0)\, e^{-i\omega t} \quad + \quad Ae^{st}$$

Ballistic term Collective mode

Collective Mode
Dispersion Relation

$$1 = \frac{k}{2}\chi(s) = \frac{k}{2}\int d\omega \frac{g(\omega)}{s - i\omega}$$

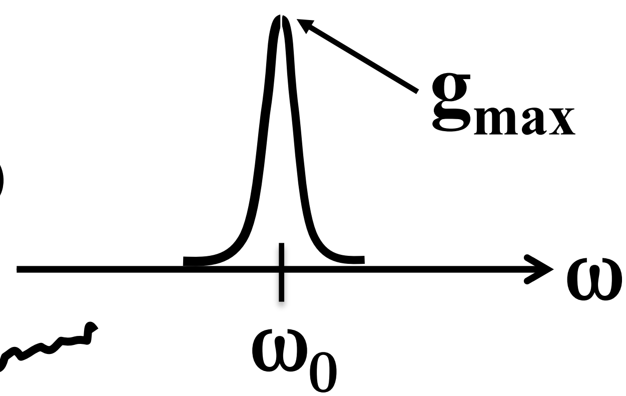
<u>S. Strogatz, Sync, p64.</u>
"Paul (Matthews) gave a lecture at the University of Warwick... One of the Professors in the audience, George Rowlands, told Paul that what we were seeing was not so strange: It's called Landau damping, and plasma physicists have known about it for 45 years."

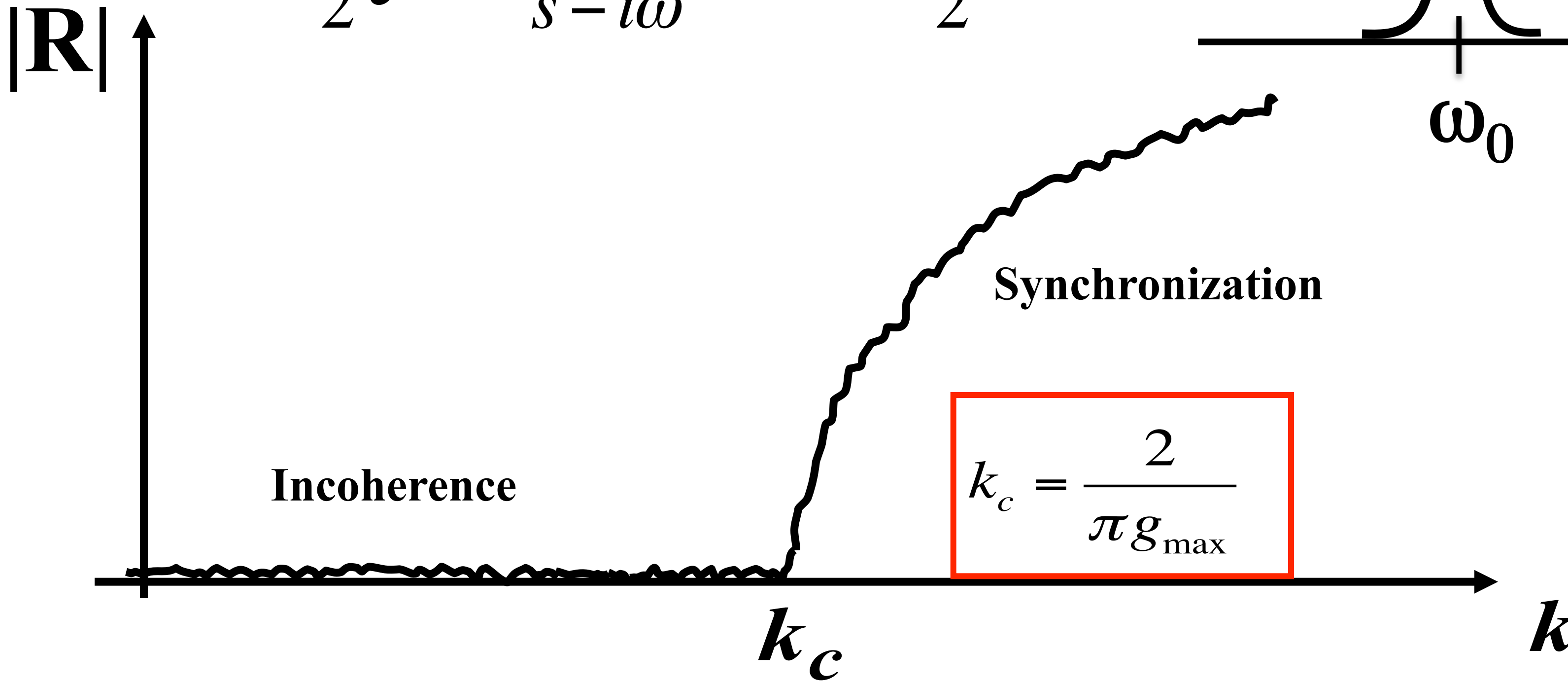
Spontaneous Synchronization

There is a transition to synchrony at a critical value of the coupling constant.

$g(\omega)$ = PDF of natural oscillator frequencies ω.

$$1 = \frac{k}{2}\int d\omega \frac{g(\omega)}{s - i\omega} \overset{s \to i\omega_0 + 0}{=} \frac{k}{2}\pi g(\omega_0)$$





What We Found

Ott & Antonsen, Chaos 18, 037113 ('08); and Chaos 19, 023119 ('09). Also Ott, Hunt & Antonsen, Chaos 19, 023117 ('09)]

- A nonlinear solution for for the distribution function in the limit N➔∞.
- The time evolution of the order parameter is that of a low dimensional system.
- The nonlinear solution is a "weak" attractor

Special case of Lorenzian $$g(\omega)=\frac{\Delta}{\pi\left[(\omega-\omega_0)^2+\Delta^2\right]}$$

$$\frac{dR}{dt}+\frac{k}{2}(|R|^2-1)R+(-i\omega_0+\Delta)R=0$$

Nonlinear Solution: the O-A Ansatz

A distribution function of the following form satisfies the kinetic equation.

$$f_{OA}(\theta,\omega,t) = \frac{g(\omega)}{2\pi}\left[\frac{1}{1-\alpha(\omega,t)e^{i\theta}} - \frac{1}{2}\right] + c.c$$

Where: $$\frac{\partial}{\partial t}\alpha + i\omega\,\alpha + \frac{k}{2}\left(R\,\alpha^2 - R^*\right) = 0$$

parametric dependence on ω

Order Parameter: $$R(t) = \int d\omega g(\omega)\,\alpha^*(\omega,t)$$

$$\frac{d}{dt}R(t) = \int d\omega g(\omega)\ \frac{\partial}{\partial t}\alpha^*(\omega,t) = \quad \ldots$$

Weak Attractor

In general, the distribution function is the sum of the O-A Ansatz and a remainder. O-A ansatz lies on a submanifold M in space of functions.

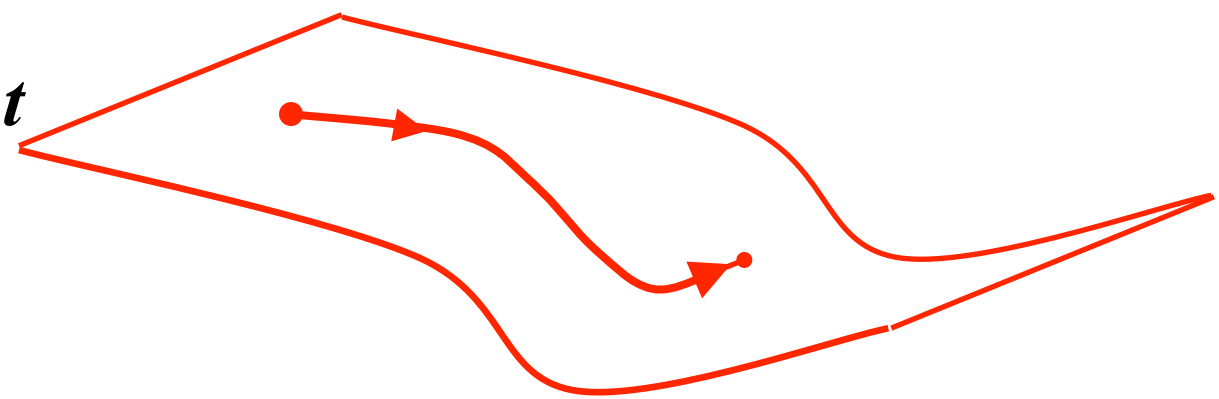
$$f(\theta,\omega,t) = f_{OA}(\theta,\omega,t) + f_{rem}(\theta,\omega,t)$$

You cannot show $\quad f_{rem}(\theta,\omega,t) \to 0 \quad \text{as} \quad t \to \infty$

You can show $\quad \iint d\omega\, d\theta\, f_{rem}(\theta,\omega,t)\, e^{-i\theta} \to 0 \quad \text{as} \quad t \to \infty$

Solution for *R(t)* is an attractor

- M **is an *invariant* submanifold.**



Generalizations of the Kuramoto Model

External Drive:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \underbrace{M_0 \sin(\Omega_0 t - \theta_i)}_{\text{drive}}$$

E.g., circadian rhythm.

Ref.: Sakaguchi, ProgTheorPhys('88); Antonsen, Faghih, Girvan, & Ott, Platig,Chaos 18 ('08); Childs, Strogatz, Chaos 18 ('08).

Communities of Oscillators:

A = # of communities; σ = community (σ = 1,2,.., s);
N_σ = # of individuals in community σ.

$$\frac{d\theta_i^\sigma}{dt} = \omega_i^\sigma + \sum_{\sigma'=1}^{s}\left(\frac{k_{\sigma\sigma'}}{N_{\sigma'}}\right)\sum_{j=1}^{N_{\sigma'}}\sin(\theta_j^{\sigma'} - \theta_i^\sigma + \beta^{\sigma\sigma'})$$

Refs, Barreto et al., PhysRevE ('08);Martens et al., PhysRevE('09); Abrams,et al.,PhysRevLett('08); Laing, Chaos19('09); and Pikovsky & Rosenblum, PhysRevLett 101 ('08).

Generalizations of the Kuramoto Model

Millennium Bridge Problem:

$$\frac{d\theta_i}{dt} = \omega_i - b\left(\frac{d^2y}{dt^2}\right)\cos(\theta_i + \beta)$$

(Walker phase)

(Bridge mode)

$$\left(\frac{d^2y}{dt^2}\right) + \nu\frac{dy}{dt} + \Omega^2 y = \frac{1}{M}\sum_i f_i$$

$$f_i(t) = f_{i0}\cos(\theta_i(t))$$

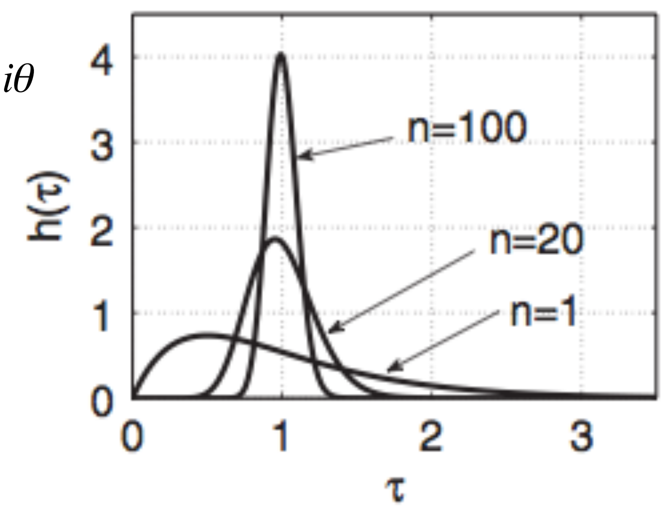
(Walker force on bridge)

Ref.: Eckhardt, Ott, Strogatz, Abrams, & McRobie, PhysRevE 75, 021110('07);
Abdulrehem and Ott, Chaos 19, 013129 ('09).

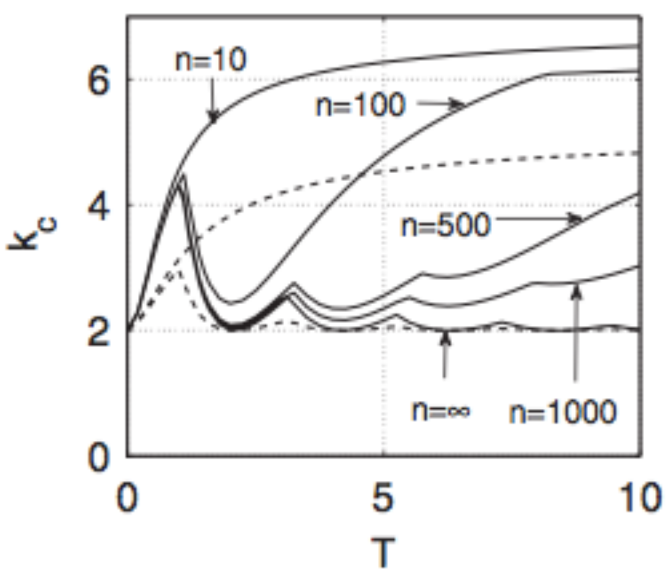
Extensions: heterogeneous time delays

W.S.Lee, E.Ott, T.M.Antonsen, *Phys.Rev.Lett.* (*'09*).

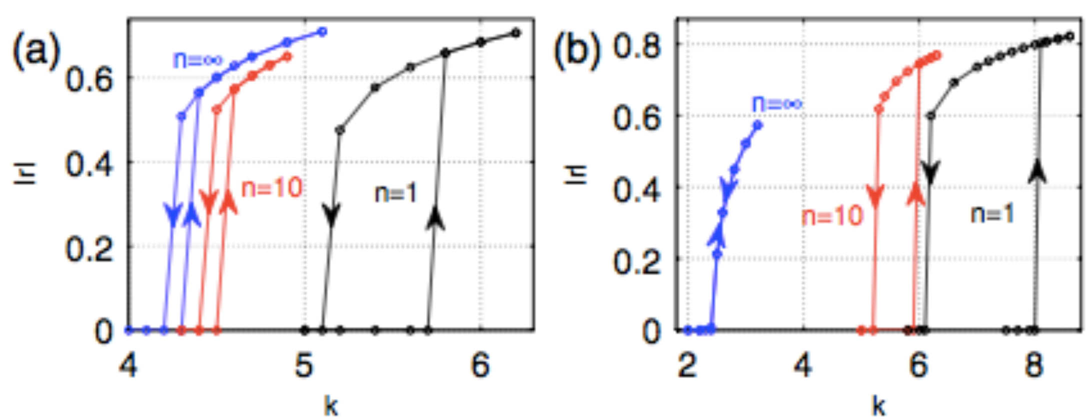
$$R(t) = \int_0^{\infty} d\tau\, h(\tau) \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega\, f(t-\tau)\, e^{i\theta}$$



Critical Coupling versus Mean Delay



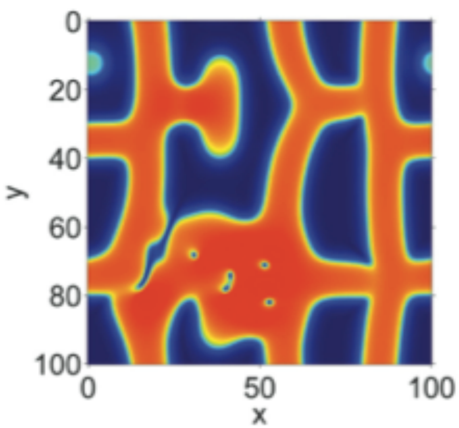
Hysteresis with heterogeneous delays



Local spatial coupling

W.S.Lee, J.G.Restrepo, E.Ott, & T.M. Antonsen, Chaos 21, 023122 (2011)

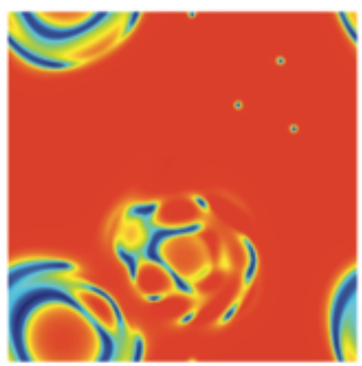
$$R(t,\boldsymbol{x}) = \int d^n x' \, q(\boldsymbol{x}-\boldsymbol{x}') \int_0^{\infty} d\tau \, h(\tau) \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega \, f(t-\tau,\boldsymbol{x}') \, e^{i\theta}$$



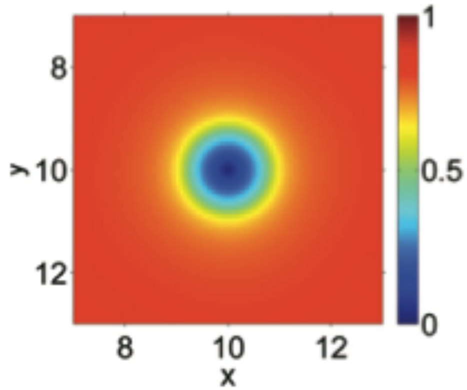
(n)t = 611

Bars

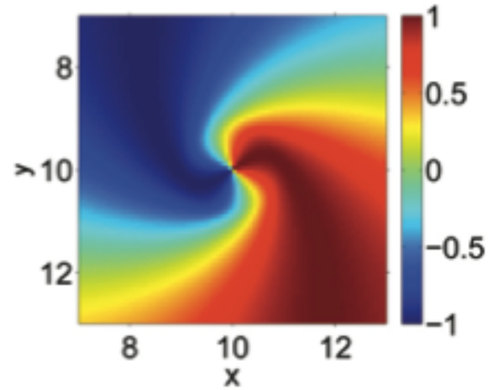
Blobs



(s) $|r|$, $t = 225$



(a)$|r(\mathbf{x})|$



(b)$\sin(\theta(\mathbf{x}))$

Spirals

Unsolved Problem: Large but finite N

TMA, E. Ott, Paul So, and Ernie Barreto

<u>Plasma approach: dressed test particles</u>

Correlations, fluctuations, and stability of a finite-size network of coupled oscillators Michael A. Buice and Carson C. Chow*, PHYSICAL REVIEW E **76**, 031118 (2007)

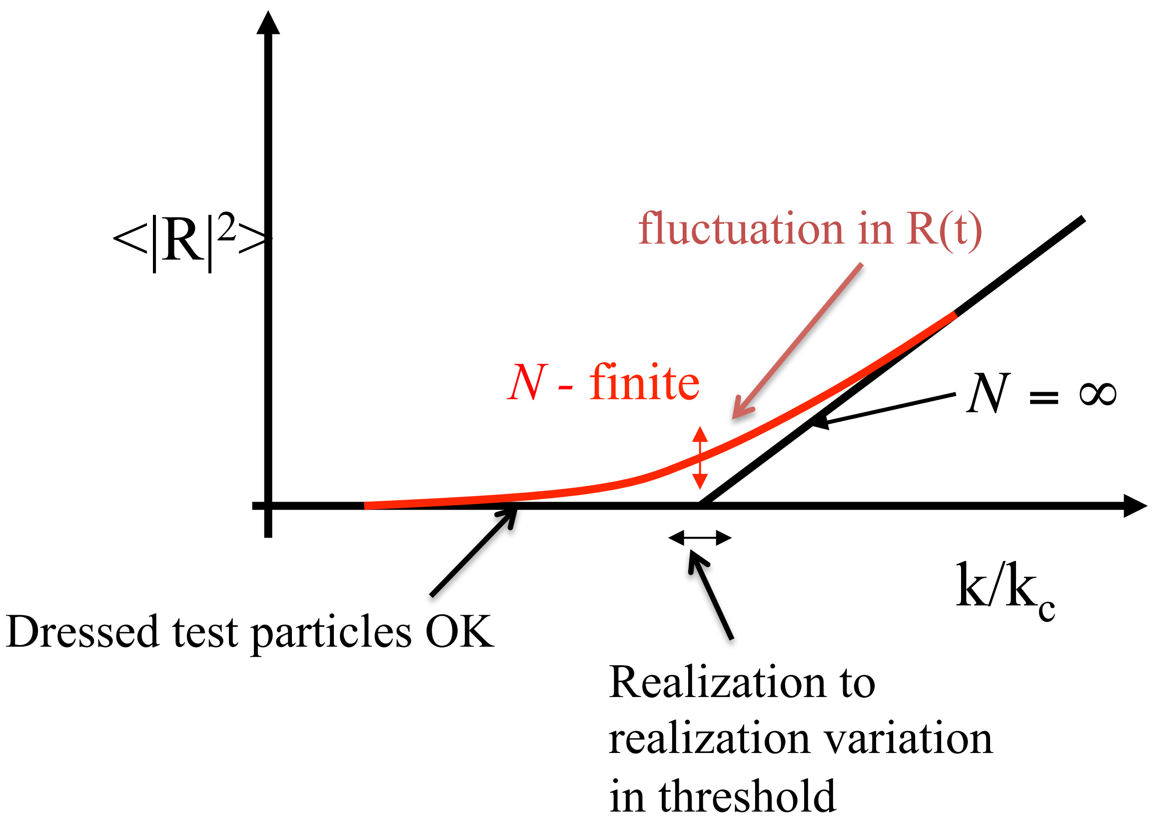
Applies to the incoherent state $k<k_c$

What's the problem?

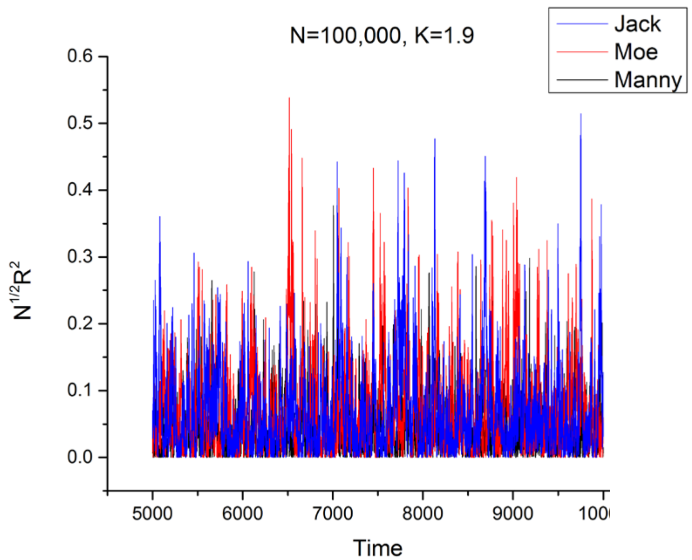
- So far no settled theory applies to the transition
  (Hong et al. PRE 2015, Daido PRE 2015 – scaling theory)
- To complicate matters: Time averaging and ensemble averaging are not the same !
- What is important? Time average of a single realization or ensemble average?

* Nat's academic brother

Fluctuations at the Transition

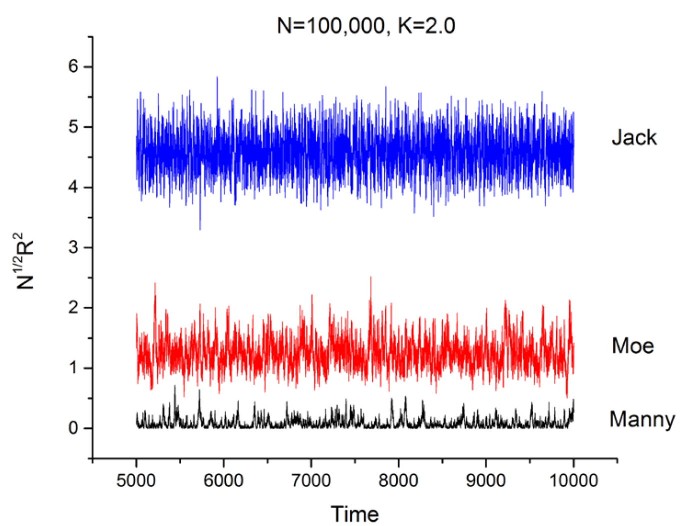


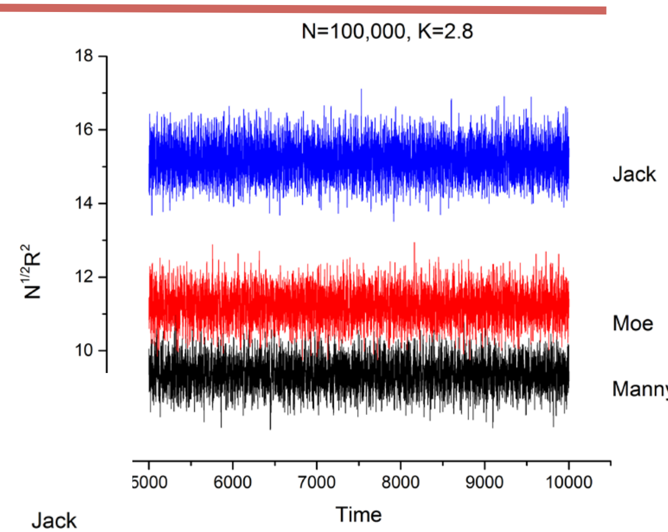
The tale of three different realizations: Manny, Mo, and Jack



k/k_c=0.85

k/k_c=1.0



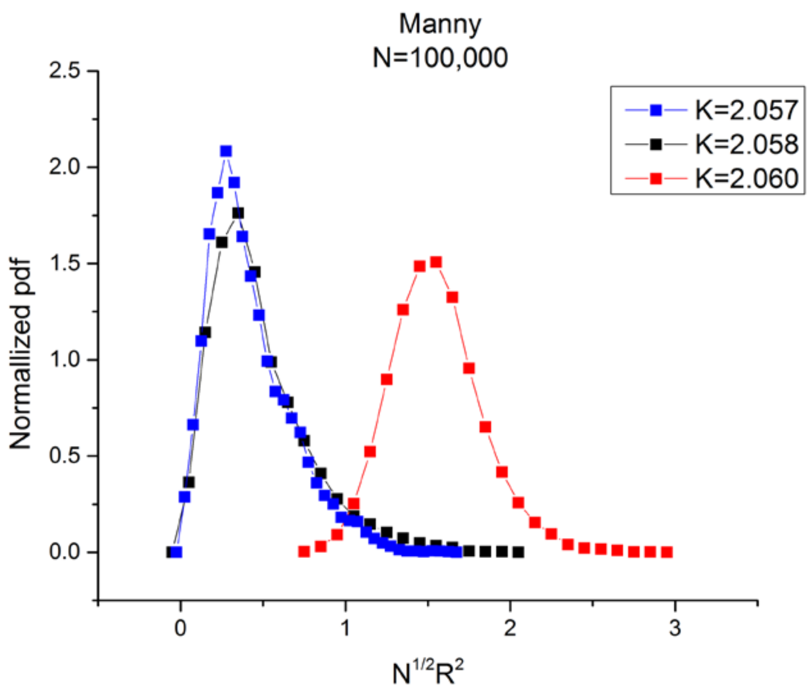


k/k_c=1.04

N = 100,000

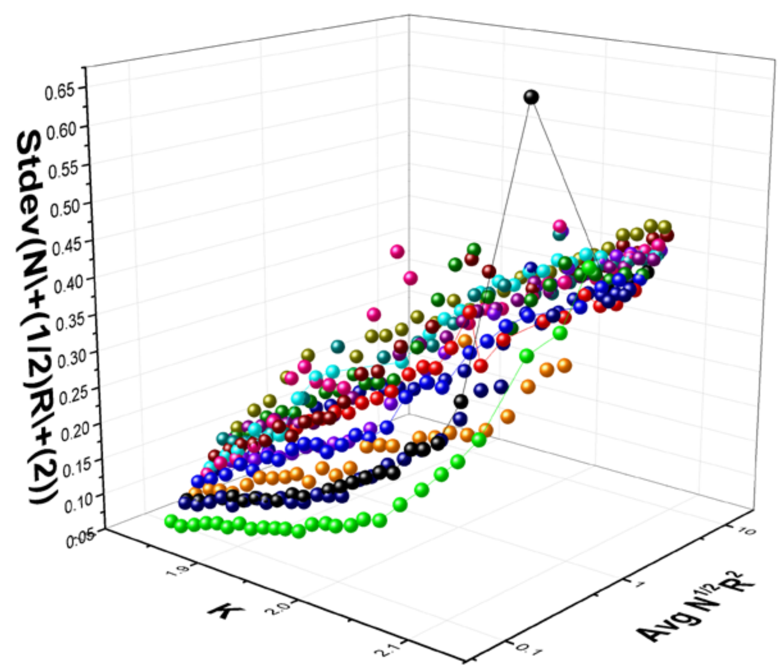
Temporal variations are only part of the story

Changes is fluctuation levels



Different realizations transition at different *k*-values.

As k is increased past k_c distribution of R^2 values evolves



Conclusion

Plasma Physics has had, and will have a big influence on the understanding and mathematical description of synchronization.

The Kuramoto model has an exact nonlinear solution in the N- infinite limit that is a weak attractor.

The correct description of the finite N with noise limit is still unresolved.

Oh yes, and Happy Birthday Nat.

Generalizations (continued)

- **Time varying link coupling strengths: P.So, B.Cotton, & E.Barreto, *Chaos 18 ('08).***

- **Distribution of heterogeneous link time delays: W.S.Lee, E.Ott, T.M.Antonsen, *Phys.Rev.Lett. ('09*).**

- **Josephson junction circuits: S.Marvel, S.Strogatz, *Chaos('09)*.**

- **Oscillators distributed in space with local coupling: This situation displays hysteresis, traveling fronts, spiral waves, target patterns, stationary spots, chimera, etc.**

  C.Laing, *Chaos 18 ('09);* and W.S.Lee, J.G.Restrepo, E.Ott, & T.M.Antonsen, Chaos **21**, 023122 (2011).

- **Birdsong model compared with experimental data on canaries: L.M.Alonso, J.A.Alliende, & G.B.Mindlin, *Europhys. Lett. ('10).***
- **And others.**