ADVANCED FOKKER-PLANCK
CALCULATIONS IN COMPLEX
MAGNETIC TOPOLOGIES

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General context

Numerical Fokker-Planck solvers are powerful tools for studying particle dynamics either for astrophysical or laboratory plasmas → extensively used in tokamaks (heating, current drive, runaway electron dynamics, …)

LUKE: 3-D linearized bounce-averaged relativistic electron Fokker-Planck solver

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The « Fokker-Planck equation »

- A Fokker–Planck equation is a deterministic equation for the time dependent probability density \( P(Y,t) \) of stochastic variables \( Y \).

- Based on general concepts (Markovian process), the Fokker-Planck equation may be applied to various domains of science, like mathematical finance.

- In plasma physics, the « Fokker–Planck » equation is the nickname of the 6-D Vlasov-Fokker-Planck equation

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x} \cdot \nabla_x f + \dot{p} \cdot \nabla_p f = C(f) + \ldots
\]

\( \dot{x} = v = p/\gamma \)  
\[ \text{(Vlasov)} \]
\( \dot{p} = e (E + v \times B) \)
\[ \{f, H\} \]

**Particle orbits**  
**Coulomb collisions**

\( O(1/\log \Lambda) \)

Small angle Coulomb collisions 
*predominate* → may be considered as a diffusion process in momentum space (Fokker-Planck operator)

Large angle (knock-on) collisions due to highly relativistic particles must be considered apart (sink/source term) → *runaway electron* avalanches

\[
C \left[ f_s, f_{s'} \right] (x, u) \equiv \frac{\Gamma_{ss'}}{2n_s} \frac{\partial}{\partial u} \int \int \int d^3 u' U (u, u') \cdot \left( \frac{\partial f_s (u)}{\partial u} \frac{f_{s'} (u')}{m_s} - \frac{f_s (u)}{m_s'} \frac{\partial f_{s'} (u')}{\partial u'} \right)
\]

Conservative form very convenient for numerical implementation.

Coulomb collisions Fokker-Planck operator

\[ \mathbf{K}_{ss'} [f_{s'}] = -\frac{\Gamma_{ss'}}{2n_s m_s} \iint d^3u' \left( \frac{\partial}{\partial u'} \cdot \mathbf{U}(u, u') \right) f_{s'} (u') \]

\[ \mathbf{D}_{ss'} [f_{s'}] = \frac{\Gamma_{ss'}}{2n_s} \iiint d^3u' \mathbf{U}(u, u') f_{s'} (u') \]

\[ \Gamma_{ss'} \equiv \frac{n_s q_s^2 e_s^2 \ln \Lambda_{ss'}}{4\pi \varepsilon_0 m_s^2} \]

**Collision kernels**

\[ \mathbf{U}(v, v') = \frac{1}{n^3} \left( w^2 \mathbb{I} - w w \right) \quad w \equiv v - v' \]

\[ \mathbf{U}(u, u') = \frac{r^2}{\gamma \gamma' w^3} \left( w^2 \mathbb{I} - u u - u' u' + r (u u' + u' u) \right) \]

*Linearized collision operator*

\[ f \simeq f_M + \delta f \]

\[ C(f, f) \simeq C(f, f_M) + C(f_M, f) \]

\[ C(f_M, f) \simeq C \left( f_M, \frac{3}{2} \xi f(M=1) (t, X, p) \right) \]

Non-relativistic  
(Rosenbluth potentials)

Relativistic  
(Braams-Karney potentials)
First principle modeling with Fokker-Planck codes

From moments of the distribution functions ($J\|, P, fast\ electron\ bremsstrahlung, ECE,...$), quantitative comparisons between modeling and experiments

$$J\| (x) = q_e \int \int \int d^3p v\| f (x, p) \quad P_{abs}^O (x) = \left. \frac{\partial \varepsilon}{\partial t} \right|_O = \int d^3p m_e c^2 (\gamma - 1) \left. \frac{\partial f (x, p)}{\partial t} \right|_O$$

Bremsstrahlung

Lower Hybrid wave current drive


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First principle modeling with Fokker-Planck codes

Lower Hybrid wave current drive

FAM

Exp.  LUKE

CR (ms⁻¹); k = 50–110 keV

cord #

Exp.  LUKE

PAM

Tore Supra


Electron cyclotron wave power absorption

TCV


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As part of a complex chain of codes, numerical Fokker-Planck solvers must be fast, accurate and robust while being able to describe multiple physical processes in complex magnetic topologies (synergistic effects).

Numerical challenge due to the large number of dimensions

3 (real space) + 3 (velocity space)

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**Particle tracking codes**  
(mostly for ion physics)

+ realistic magnetic configurations (3D)  
+ very complex particle orbits (*potatoe*)  
- difficulty to cover full energy range  
  (*bulk ↔ tail*)  
- Noisy particle distribution (statistical)  
  (*derivatives, moments*)  
- intrinsically slow code  
+ low memory consumption  
+ well designed for parallel processing

**Finite difference codes**  
(mostly for electron physics)

- simplified magnetic configurations (2D)  
- restricted types of particle orbits  
  (*trapped/passing*)  
+ full energy range (*bulk + tail*)  
+ smooth particle distributions (*moments*)  
+ intrinsically fast code  
  (*reverse time scheme*)  
- high memory consumption (*large matrix*)  
- parallel processing possible  
  (*linear system solver*)

**SPOT, NEMO, TRANSP, ASCOT, …**

**CQL3D (e/i), LUKE (e),…**

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Ion vs electron trajectories

$\alpha$ particle (3.5 MeV) in ITER


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The number of dimensions must be reduced by appropriate averages (geometrical symmetry, physics ordering) → *FP equation may be solved by existing computers*

- 2-D magnetic configuration, *toroidal axisymmetry* → $\phi$ toroidal angle averaging $(6$-D $\to 5$-D)
- The magnetic field line is a local axis of symmetry in velocity space → $\varphi$ gyroangle averaging, *guiding-center physics* $(5$-D $\to 4$-D)
- *Low collisionality limit*, poloidal orbits are complete before particles are scattered-off by Coulomb collisions → $\theta$ poloidal angle averaging $(4$-D $\to 3$-D)

- Nested poloidal magnetic flux surfaces with a single minimum of B for each $\psi$ → *two types of particle orbits: trapped and passing.*
- Small parameter $\delta$ expansion of the distribution $f = f_0 + f_1 + f_2 + \ldots$

$$
\delta^2 \simeq \frac{\rho}{R} \simeq \frac{t_b}{\Omega} \quad \frac{\nu}{\Omega} \leq \delta^2
$$
Electron Fokker-Planck equation for the zero order distribution function $f_0$

$$\frac{\partial f_0}{\partial t} + \mathbf{v}_{cg} \cdot \nabla_{\mathbf{x}_R} f_0$$

$$= C(f_0) + Q(f_0) + T(f_0) + E(f_0)$$

- $E(f_0) = -\nabla_{\mathbf{p}} \left( e \left\langle \mathbf{E}_2 \right\rangle_{\Omega, \omega} \cdot f_0 \right)$
  - electric field
- $Q(f_0) \equiv \nabla_{\mathbf{p}} \cdot (\mathbb{D}_{ql} \cdot \nabla_{\mathbf{p}} f_0)$
  - rf waves
- $T(f_0) \equiv \nabla_{\mathbf{x}_T} \cdot (\mathbb{D}_{\mathbf{x}} \cdot \nabla_{\mathbf{x}_T} f_0)$
  - radial transport

$v_{cg} \simeq p || \hat{\mathbf{b}} / \gamma + \mathbf{v}_D$ 
- drift velocity

$p = p || \hat{\mathbf{b}} + p_\perp$

Bounce-averaged electron Fokker-Planck equation

low collisionality $\rightarrow \delta^2 \ll \nu^* \ll 1$

zero-banana width approximation $\rightarrow \mathbf{V}_D = 0$

$$\{ \mathcal{O} \} \equiv \frac{1}{\chi q} \left[ \frac{1}{2} \sum_{\sigma} T \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \frac{d\theta}{2\pi} \frac{1}{|\hat{\psi} \cdot \hat{r}|} \frac{r}{a_p} \frac{B}{B_P} \frac{\xi_0}{\xi} \mathcal{O} \right]$$

$$f^{(0)} (\psi, p, \xi_0) = \{ f_0 (\psi, \theta, p, \xi) \}$$

$$\{ \mathbf{v}_{cg} \cdot \nabla \mathbf{x}_R f_0 \} = 0$$

Very large advection term annihilated $\rightarrow$ Fokker-Planck equation well-conditioned by collisions.

$$\partial \{ f_0 \} / \partial t = \{ C (f_0) \} + \{ Q (f_0) \} + \{ E (f_0) \}$$

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From particle dynamics projected at $B = B_{\text{min}}$

\[ f(\psi, \theta, p, \xi) = f^{(0)}(\psi, p, \xi_0) \]

\[ \xi = \sigma \sqrt{1 - \Psi(\psi, \theta)(1 - \xi_0^2)} \]


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Conservative form of the bounce-averaged electron Fokker-Planck equation

\[ \frac{\partial f^{(0)}}{\partial t} + \nabla \cdot S^{(0)} = s_{+}^{(0)} - s_{-}^{(0)} \]

\[ \nabla \cdot S^{(0)} = \frac{B_0}{\tilde{q}\lambda} \frac{\partial}{\partial \psi} \left( \tilde{q}\lambda \| \nabla \psi \| S_{\psi}^{(0)} \right) + \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 S_{p}^{(0)} \right) - \frac{1}{\lambda p} \frac{\partial}{\partial \xi_0} \left( \lambda \sqrt{1 - \xi_0^2} S_{\xi}^{(0)} \right) \]

\[ S^{(0)} = -\mathcal{D}^{(0)} \cdot \nabla f^{(0)} + \mathbf{F}^{(0)} f^{(0)} \]

Implicit numerical formulation of the Fokker-Planck equation

\[
\left( \frac{\hat{A}}{\Delta t} + \hat{B} \right) X^{(k+1)} = \left( \frac{\hat{A}}{\Delta t} \right) X^{(k)} + \hat{C} \left( X_M, X^{(k)} \right) + \hat{S}_R^{(k)}
\]

Cross-derivatives consistent with boundary conditions are critically important to keep the numerical solution conservative (density)

~ $10^6 \times 10^6$ entries, highly sparse matrix

Best performances with multifrontal massively parallel sparse direct solver (MUMPS)


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Beyond standard finite-difference Fokker-Planck calculations

The challenge is to describe numerically more complex kinetic problems, *keeping the advantages of smooth distributions (finite difference, …)* with high computational performances for integrated modeling (parallel processing).

1) **finite-banana width effects** → *bootstrap current, Ware pinch, rf wave induced transport,… for non-Maxwellian distributions, 3-D problem*

2) high collisionality regime (SOL) → *4-D/5-D gyrokinetic problem*

3) 3-D magnetic configurations (MHD, turbulence, magnetic ripple, other magnetic configuration) → *5-D problem (guiding-center)*

Finite-banana width effects have been investigated numerically for non-Maxwellian distributions in the thin banana width limit → **small drift approximation**.

\[ \mathbf{v}_{cg} \simeq p_{||} \hat{\mathbf{b}}/\gamma + \mathbf{v}_D \quad \| \mathbf{v}_D \| / \| \mathbf{v}_{cg} \| \ll 1 \]

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Drift kinetic Fokker-Planck equations and bootstrap current calculations with rf waves

\[
\frac{\partial \{f_0\}}{\partial t} = \{C(f_0)\} + \{Q(f_0)\} + \{E(f_0)\}
\]

\[
\tilde{f} = \frac{v_\parallel}{\Omega_e} I(\psi) \frac{\partial f_0}{\partial \psi}
\]

\[
\{C(g)\} + \{Q(g)\} + \{E(g)\} = -\{C(\tilde{f})\} - \{Q(\tilde{f})\} - \{E(\tilde{f})\}
\]

\[
f = f_0 + f_1 = f_0 + \tilde{f} + g
\]

4 Fokker-Planck equations to be solved for 1 effective radial position

Used for rf current drive in ITB’s

Finite banana width effects from an exact guiding-center Hamiltonian

Derive an exact guiding-center Hamiltonian and Poisson bracket (Lie transform)

\[
\frac{\partial f_{gc}}{\partial t} + \{f_{gc}, H_{gc}\}_{gc} = C_{gc} \{f_{gc}, H_{gc}\}
\]

\[
C_{gc} [f] = -\frac{1}{J_{gc}} \frac{\partial}{\partial Z^\alpha} \left[ J_{gc} \left( K_{gc}^\alpha f - D_{gc}^{\alpha\beta} \frac{\partial f}{\partial Z^\beta} \right) \right]
\]

\[
\langle \ldots \rangle_{O} \equiv \frac{1}{\tau_{O}} \int_{\mathcal{O}} d\theta \frac{d\theta}{\dot{\theta}} \quad \text{orbit averaging (num.)}
\]

\[
C^{(0)}_{gc} \left[ f^{(0)} \right] = -\frac{1}{J_{\mathcal{O}}} \frac{\partial}{\partial I^a} \left[ J_{\mathcal{O}} \left( K_{gc}^{a(0)} f^{(0)} - D_{gc}^{ab(0)} \frac{\partial f^{(0)}}{\partial I^b} \right) \right] + M_{gc}^{(0)} f^{(0)}
\]

divergence form

Convection vector and diffusion tensor in the thin orbit approximation

\[
\begin{align*}
\mathbf{K}^{\psi}_{gc} &= \epsilon_{\psi} \delta_{\psi} \left[ \nu_l + \frac{1}{\xi} \nu_t \right] + \mathcal{O} \left( \epsilon^2, \epsilon_{\psi}, \epsilon_{\psi}^2 \right) \\
\mathbf{K}^{p}_{gc} &= - \left[ 1 - \epsilon_{\lambda_{gc}} \frac{1 - \xi^2}{2\xi} \frac{\partial}{\partial \xi} \right] \nu_p + \mathcal{O} \left( \epsilon^2, \epsilon_{\psi}, \epsilon_{\psi}^2 \right) \\
\mathbf{K}^{x}_{gc} &= - \frac{1 - \xi^2}{2\xi} \left[ (\epsilon_{\psi} \delta_{\psi} + \epsilon_{\lambda_{gc}}) \nu_l + \frac{2\xi}{1 - \xi^2} \left( 1 - \epsilon_{\lambda_{gc}} - \frac{1 - \xi^2}{2\xi} \frac{\partial}{\partial \xi} + \epsilon_{\psi} \delta_{\psi} \frac{1 - \xi^2}{2\xi^2} \right) \nu_t \right] + \mathcal{O} \left( \epsilon^2, \epsilon_{\psi}, \epsilon_{\psi}^2 \right)
\end{align*}
\]

Thin orbit approximation: \( \epsilon_{\psi} \equiv (\psi - \overline{\psi}) / \overline{\psi} \ll 1 \)

Magnetic non-uniformity: \( \epsilon \equiv \rho / L_B \ll 1 \)

Classical Maxwellian bootstrap current recovered analytically (Lorentz collision operator)

LUKE 2: numerical algorithm of LUKE preserved, but more off-diagonals elements, Jacobian modified, and the physical meaning of \( \xi \) is no more a cosine of the pitch-angle

Solved and unsolved problems for Fokker-Planck calculations in tokamaks

Particle tracking codes (anomalous orbits)

3-D finite-difference codes (zero-banana width). *Finite-orbit width effects soon implemented for LUKE code (regular orbits)*

4-D/5-D gyrokinetic FEM code for SOL

5-D guiding-center calculations (finite-difference method)

\[ t > t_{\text{toroidal}} \]