DE LA RECHERCHE À L'INDUSTRIE

# ADVANCED FOKKER-PLANCK CALCULATIONS IN COMPLEX MAGNETIC TOPOLOGIES

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*Symposium in honor of Pr. Nathaniel J.Fisch*

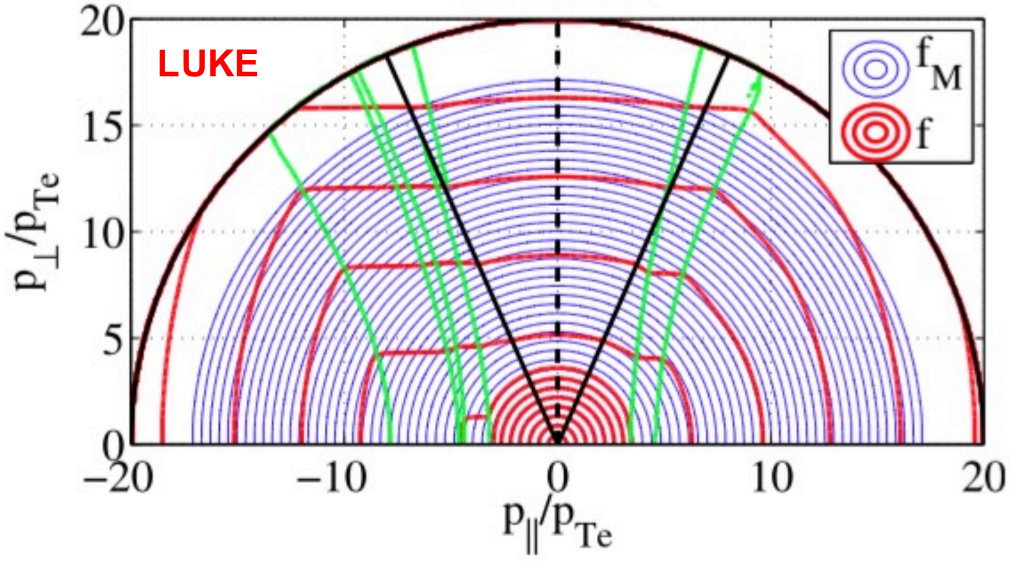
www.cea.fr

*Acknowledgements to M. Shoucri, J. Decker and A. Brizard*

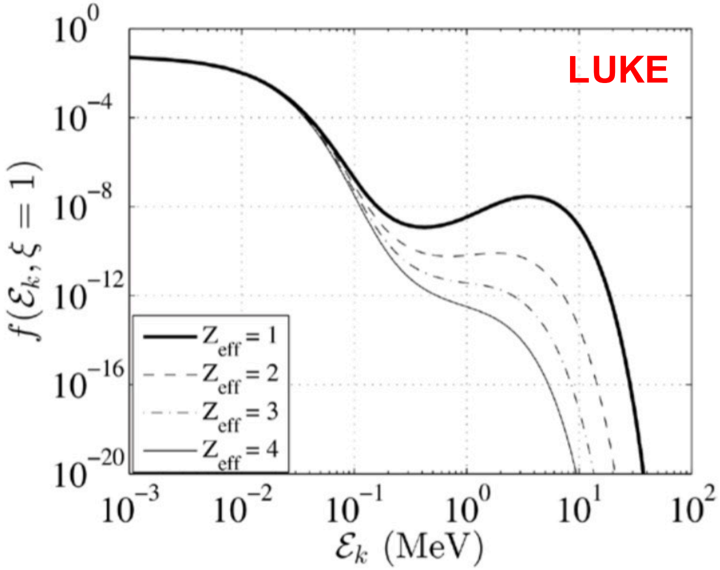
# General context

Numerical Fokker-Planck solvers are powerful tools for studying particle dynamics either for astrophysical or laboratory plasmas → *extensively used in tokamaks (heating, current drive, runaway electron dynamics,…)*

Lower Hybrid wave current drive




Runaway electrons (ALD force)




**LUKE**: 3-D linearized bounce-averaged relativistic electron Fokker-Planck solver

J. Decker, Y. Peysson et al., *Phys. Plasmas* 21, 092504 (2014)

J. Decker, et al., *Plasma Phys. Contr. Fusion* 58, 025016 (2016)

# The « Fokker-Planck equation »

- A Fokker–Planck equation is a deterministic equation for the time dependent probability density P(Y,t) of stochastic variables Y.

- Based on general concepts (Markovian process), the Fokker-Planck equation may be applied to various domains of science, like mathematical finance.

- In plasma physics, the « Fokker–Planck » equation is the nickname of the 6-D Vlasov-Fokker-Planck equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f}_{\text{Particle orbits}} = \underbrace{C(f)}_{\text{Coulomb collisions}} + \ldots$$

$$\dot{\mathbf{x}} = \mathbf{v} = \mathbf{p}/\gamma$$

$$\dot{\mathbf{p}} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

*Particle orbits*

(Vlasov)

$\{f, H\}$

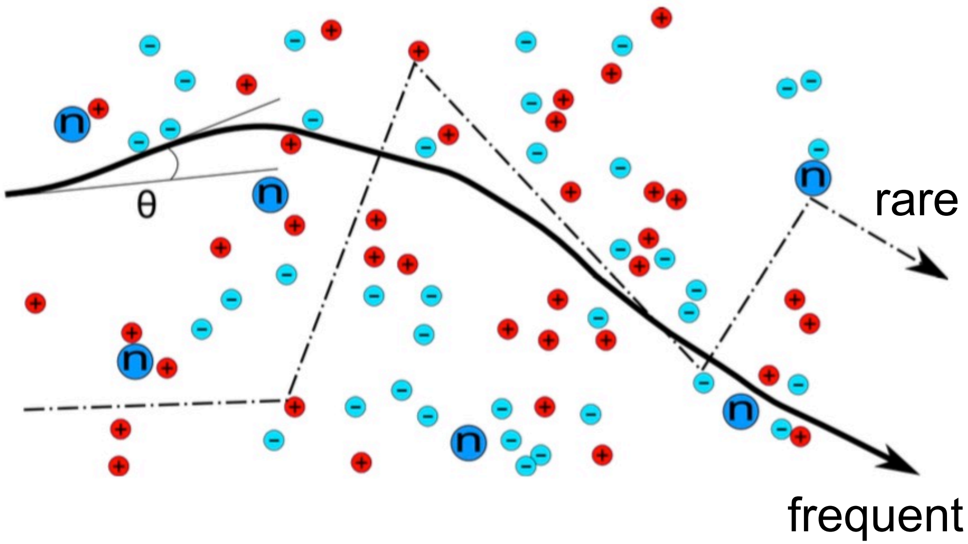
Coulomb collisions

(**Fokker-Planck**)

$\mathcal{O}\left(1/\log\Lambda\right)$

Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)

# Coulomb collisions Fokker-Planck operator




- *Small angle Coulomb collisions predominate* → may be considered as a *diffusion process* in momentum space (**Fokker-Planck operator**)
- Large angle (knock-on) collisions due to highly relativistic particles must be considered apart (sink/source term) → *<u>runaway electron</u> avalanches*

$$C\left[f_s, f_{s'}\right](\mathbf{x}, \mathbf{u}) \equiv \frac{\Gamma_{ss'}}{2n_s}\frac{\partial}{\partial \mathbf{u}} \cdot \iiint \mathrm{d}^3\mathbf{u}' \mathbb{U}\left(\mathbf{u}, \mathbf{u}'\right) \cdot \left(\frac{\partial f_s\left(\mathbf{u}\right)}{\partial \mathbf{u}}\frac{f_{s'}\left(\mathbf{u}'\right)}{m_s} - \frac{f_s\left(\mathbf{u}\right)}{m_{s'}}\frac{\partial f_{s'}\left(\mathbf{u}'\right)}{\partial \mathbf{u}'}\right)$$

$$C\left[f_s, f_{s'}\right] = -\frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{K}_{ss'}\left[f_{s'}\right] f_s - \mathbb{D}_{ss'}\left[f_{s'}\right] \cdot \frac{\partial f_s}{\partial \mathbf{p}}\right)$$

***Conservative form very convenient for numerical implementation.***

E. Nilsson, et al., *Plasma Phys. Contr. Fusion* 57, 095006 (2015)

# Coulomb collisions Fokker-Planck operator

$$\mathbf{K}_{ss'}\left[f_{s'}\right] = -\frac{\Gamma_{ss'}}{2n_s}\frac{m_s}{m_{s'}}\iiint \mathrm{d}^3\mathbf{u}'\left(\frac{\partial}{\partial \mathbf{u}'}\cdot \mathbb{U}\left(\mathbf{u},\mathbf{u}'\right)\right) f_{s'}\left(\mathbf{u}'\right)$$ *convection*

$$\mathbb{D}_{ss'}\left[f_{s'}\right] = \frac{\Gamma_{ss'}}{2n_s}\iiint \mathrm{d}^3\mathbf{u}'\mathbb{U}\left(\mathbf{u},\mathbf{u}'\right) f_{s'}\left(\mathbf{u}'\right)$$ *diffusion*

$$\Gamma_{ss'} \equiv \frac{n_s q_s^2 e_{s'}^2 \ln \Lambda_{ss'}}{4\pi\varepsilon_0 m_s^2}$$

***Collision kernels***

$$\mathbb{U}\left(\mathbf{v},\mathbf{v}'\right) = \frac{1}{w^3}\left(w^2\mathbb{I} - \mathbf{w}\mathbf{w}\right) \qquad \mathbf{w} \equiv \mathbf{v} - \mathbf{v}'$$

$$\mathbb{U}\left(\mathbf{u},\mathbf{u}'\right) = \frac{r^2}{\gamma\gamma' w^3}\left(w^2\mathbb{I} - \mathbf{u}\mathbf{u} - \mathbf{u}'\mathbf{u}' + r\left(\mathbf{u}\mathbf{u}' + \mathbf{u}'\mathbf{u}\right)\right)$$

Non-relativistic

(Rosenbluth potentials)

Relativistic

(Braams-Karney potentials)

*Linearized collision operator*

$$\boxed{f \simeq f_M + \delta f}$$

$$C\left(f,f\right) \simeq C\left(f,f_M\right) + C\left(f_M,f\right)$$

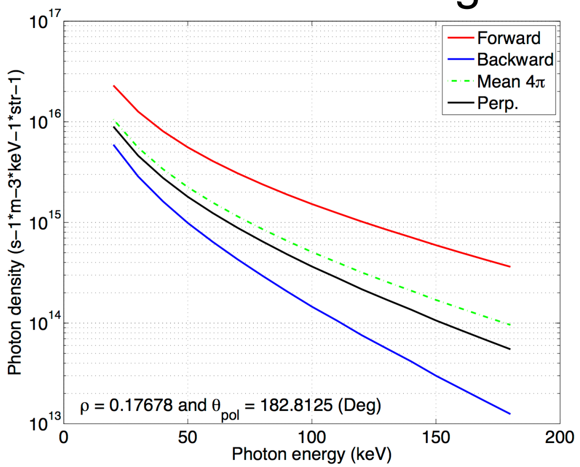
$$C\left(f_M,f\right) \simeq C\left(f_M, \frac{3}{2}\xi f^{(m=1)}\left(t,\mathbf{X},p\right)\right)$$

# First principle modeling with Fokker-Planck codes

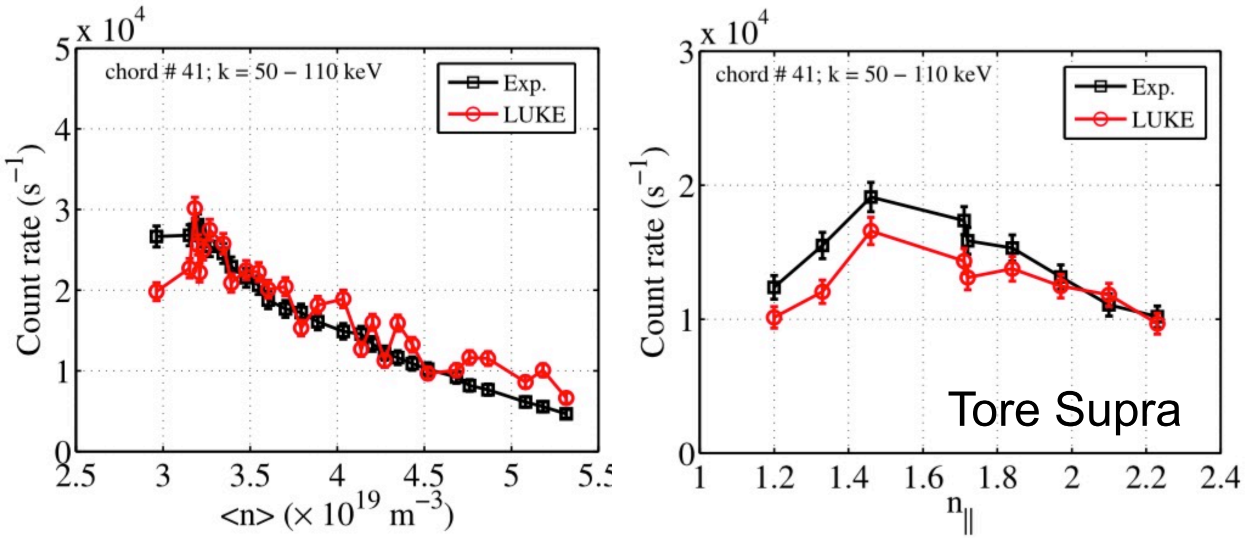
From moments of the distribution functions *($J_{\|}$, P, fast electron bremsstrahlung, ECE,…)*, quantitative comparisons between modeling and experiments

$$J_{\|}(\mathbf{x}) = q_e \iiint d^3p\, v_{\|} f(\mathbf{x},\mathbf{p}) \qquad P_{abs}^{\mathcal{O}}(\mathbf{x}) = \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathcal{O}} = \int d^3p\, m_e c^2 (\gamma - 1) \left.\frac{\partial f(\mathbf{x},\,\mathbf{p})}{\partial t}\right|_{\mathcal{O}}$$

Bremsstrahlung




Lower Hybrid wave current drive

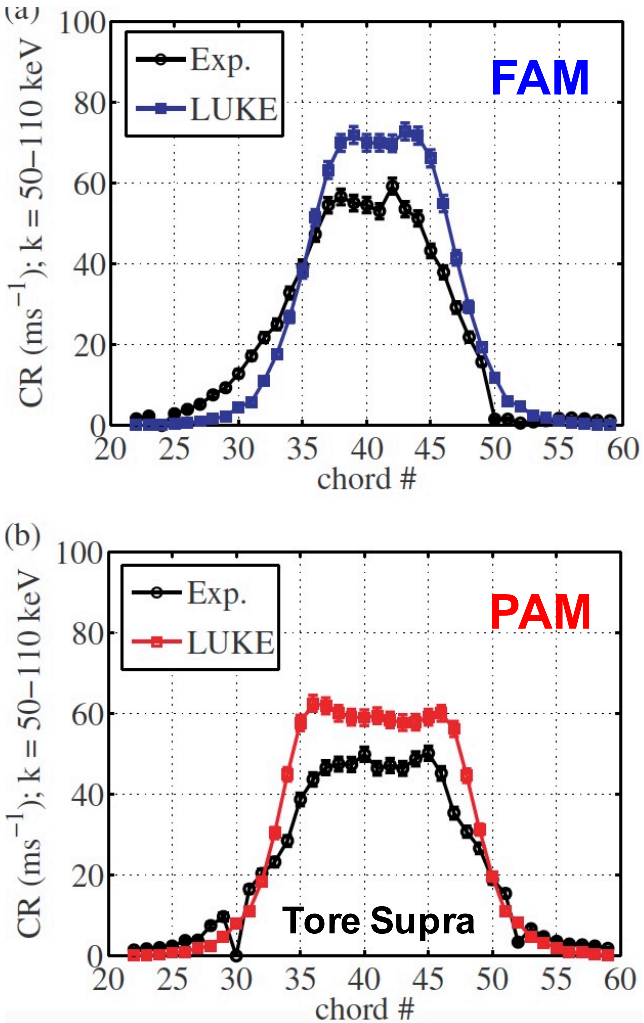



Y. Peysson and J. Decker, *Phys. Plasmas* 15, 092509 (2008)

J. Decker, Y. Peysson et al., *Phys. Plasmas* 21, 092504 (2014)

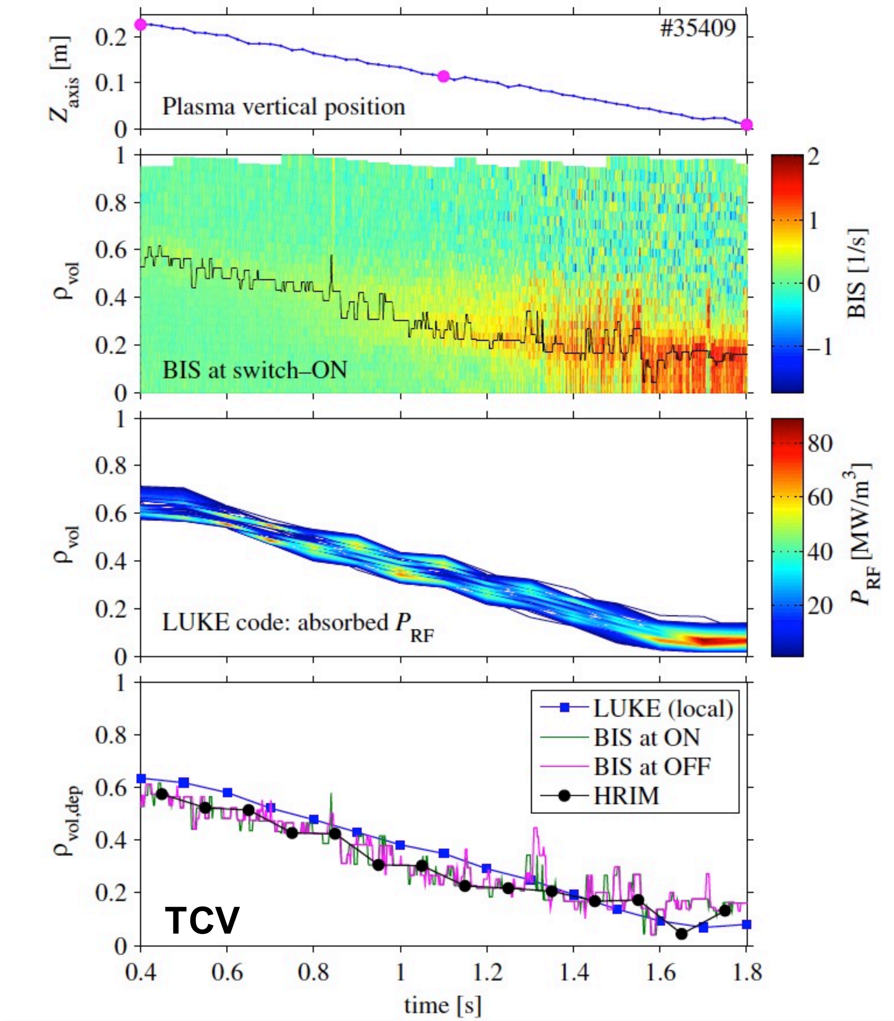
# First principle modeling with Fokker-Planck codes

Lower Hybrid wave current drive




E. Nilsson, et al., *Nucl. Fusion* 53, 083018 (2013)

Electron cyclotron wave power absorption

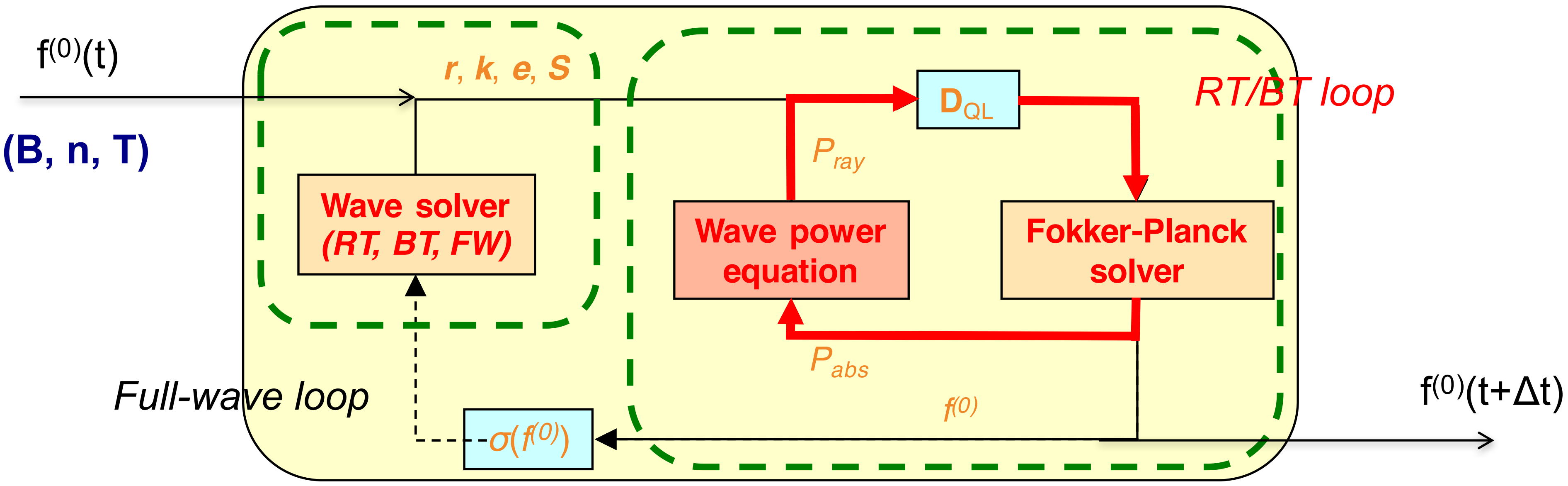



L. Curchod, et al., *Plasma Phys. Contr. Fusion* 53, 115005 (2011)

# Fokker-Planck codes and integrated modeling

As part of a complex chain of codes, numerical Fokker-Planck solvers must be **fast**, **accurate** and **robust** while being able to describe multiple physical processes in complex magnetic topologies (synergistic effects).

***heating/current drive module***




Numerical challenge due to the large number of dimensions
**3 (real space) + 3 (velocity space)**

A. Bécoulet et al, *Comp. Phys. Comm.* 177, 55 (2007)
Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)

# Fokker-Planck code types

## Particle tracking codes
*(mostly for ion physics)*

+ realistic magnetic configurations (3D)
+ very complex particle orbits *(potatoe)*
- difficulty to cover full energy range *(bulk ↔ tail)*
- Noisy particle distribution (statistical) *(derivatives, moments)*
- intrinsically slow code
+ low memory consumption
+ well designed for parallel processing

**SPOT, NEMO, TRANSP, ASCOT, …**

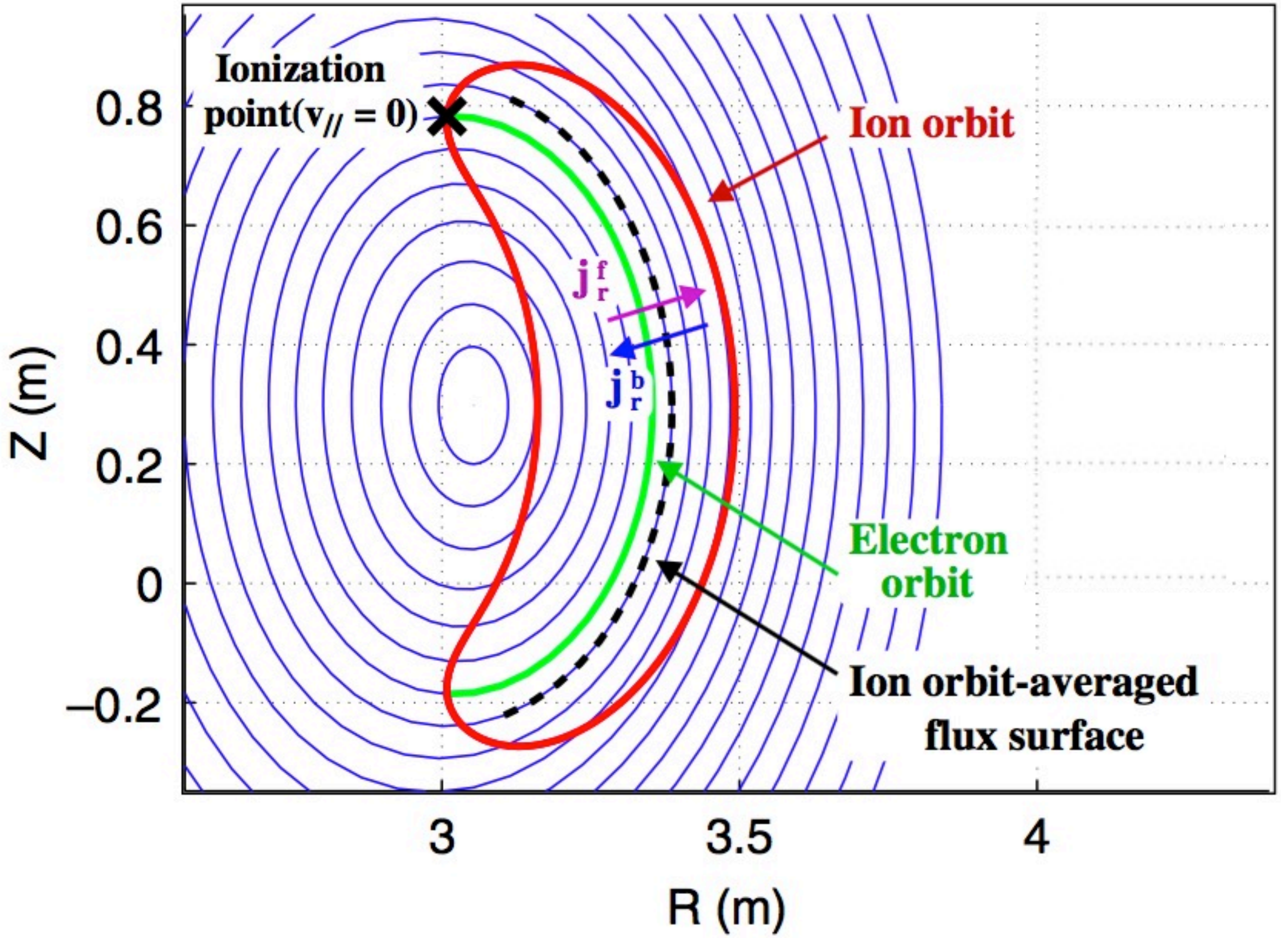
## Finite difference codes
*(mostly for electron physics)*

- simplified magnetic configurations (2D)
- restricted types of particle orbits *(trapped/passing)*
+ full energy range *(bulk + tail)*
+ smooth particle distributions *(moments)*
+ intrinsically fast code *(reverse time scheme)*
- high memory consumption *(large matrix)*
- parallel processing possible *(linear system solver)*

**CQL3D (e/i), LUKE (e),…**

# Ion vs electron trajectories

**α particle (3.5 MeV) in ITER**




M. Schneider, et al., *Plasma Phys. Contr. Fusion* 47, 2087 (2005)

# Finite-difference bounce-averaged electron Fokker-Planck solvers

The number of dimensions must be reduced by appropriate averages (geometrical symmetry, physics ordering) → *FP equation may be solved by existing computers*

- 2-D magnetic configuration, *toroidal axisymmetry* → $\phi$ toroidal angle averaging **(6-D → 5-D)**
- The magnetic field line is a local axis of symmetry in velocity space → $\varphi$ gyroangle averaging, *guiding-center physics* **(5-D → 4-D)**
- *Low collisionality limit,* poloidal orbits are complete before particles are scattered-off by Coulomb collisions → $\theta$ poloidal angle averaging **(4-D → 3-D)**

- Nested poloidal magnetic flux surfaces with a single minimum of B for each $\psi$ → *two types of particle orbits: trapped and passing*.
- Small parameter $\delta$ expansion of the distribution $f = f_0 + f_1 + f_2 + \ldots$

$$\delta^2 \simeq \rho/R \simeq t_b/\Omega \qquad\qquad \nu/\Omega \leq \delta^2$$

Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)

Y. Peysson and J. Decker, AIP Conf. Proc., 1069, 176 (2008)

# Electron Fokker-Planck equation for the zero order distribution function $f_0$

$$\partial f_0/\partial t + \mathbf{v}_{cg}\cdot\nabla_{\mathbf{x}_R} f_0 = C(f_0) + Q(f_0) + T(f_0) + E(f_0)$$

$$\begin{cases} E(f_0) = -\nabla_{\mathbf{p}}\left(e\left\langle \mathbf{E}_2\right\rangle_{\Omega,\omega}\cdot f_0\right) & \textit{electric field} \\ Q(f_0) \equiv \nabla_{\mathbf{p}}\cdot\left(\mathbb{D}_{ql}\cdot\nabla_{\mathbf{p}} f_0\right) \longrightarrow \boxed{\mathbb{D}_{ql} \propto \left|\left|\mathbf{E}_1\right|\right|^2} & \textit{rf waves} \\ T(f_0) \equiv \nabla_{\mathbf{x}_T}\cdot\left(\mathbb{D}_{\mathbf{x}}\cdot\nabla_{\mathbf{x}_T} f_0\right) & \textit{radial transport} \end{cases}$$

$$\mathbf{v}_{cg} \simeq p_{\parallel}\hat{\mathbf{b}}/\gamma + \boxed{\mathbf{v}_D} \quad \text{drift velocity}$$

$$\mathbf{p} = p_{\parallel}\hat{\mathbf{b}} + \mathbf{p}_{\perp}$$

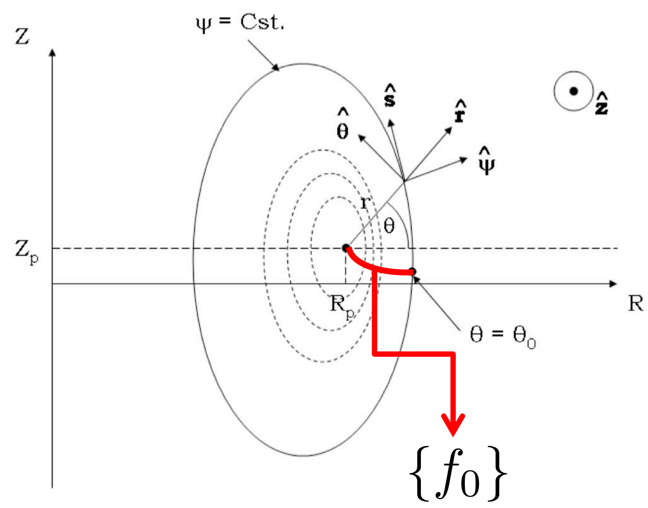
Y. Peysson and J. Decker, AIP Conf. Proc., 1069, 176 (2008)

# Bounce-averaged electron Fokker-Planck equation

low collisionality → $\delta^2 \ll \nu^* \ll 1$

zero-banana width approximation → $\mathbf{V}_D = 0$

$$\{\mathcal{O}\} \equiv \frac{1}{\lambda\tilde{q}}\left[\frac{1}{2}\sum_\sigma\right]_T \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{2\pi}\frac{1}{\left|\hat{\psi}\cdot\hat{\mathbf{r}}\right|}\frac{r}{a_p}\frac{B}{B_P}\frac{\xi_0}{\xi}\mathcal{O}$$




$$f^{(0)}(\psi,p,\xi_0) = \{f_0(\psi,\theta,p,\xi)\}$$

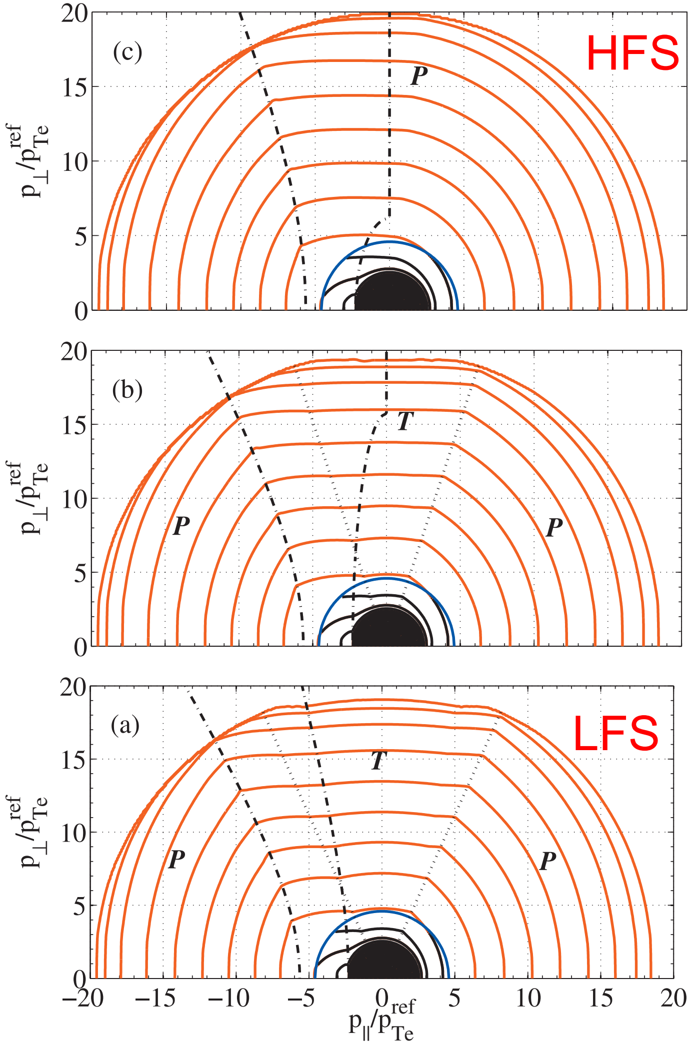
$$\{\mathbf{v}_{cg}\cdot\nabla_{\mathbf{x}_R} f_0\} = 0$$

*Very large advection term annihilated → Fokker-Plank equation well-conditionned by collisions.*

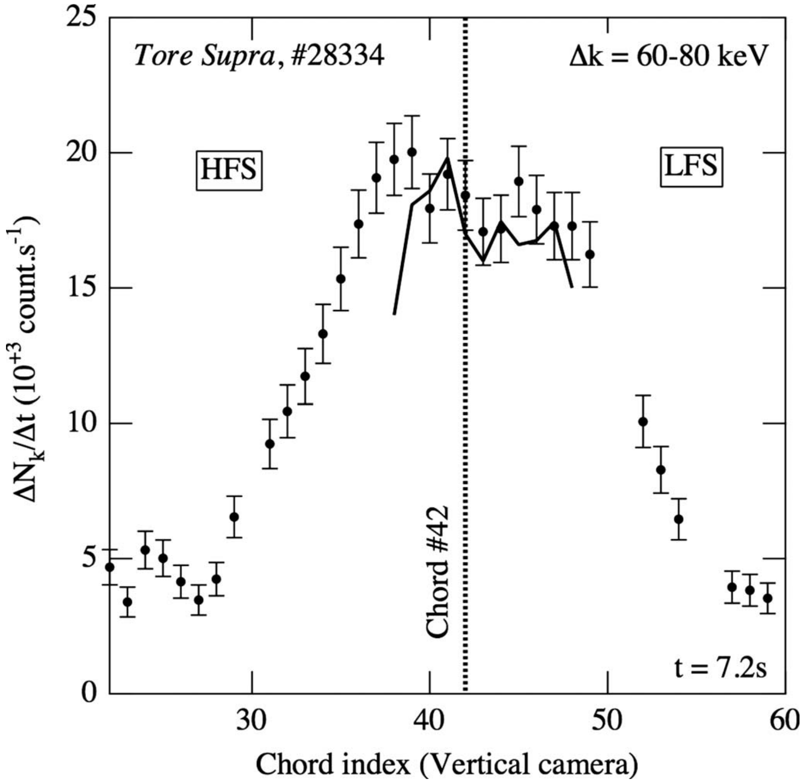
t > $t_{bounce}$

$$\partial\{f_0\}/\partial t = \{C(f_0)\} + \{Q(f_0)\} + \{E(f_0)\}$$

# From particle dynamics projected at $B = B_{min}$




Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)




$$f(\psi, \theta, p, \xi) = f^{(0)}(\psi, p, \xi_0)$$

$$\xi = \sigma\sqrt{1 - \Psi(\psi, \theta)(1 - \xi_0^2)}$$

Y. Peysson and J. Decker, *Phys. Plasmas* 15, 092509 (2008)

# Conservative form of the bounce-averaged electron Fokker-Planck equation

LUKE

$$\partial f^{(0)}/\partial t + \nabla \cdot \mathbf{S}^{(0)} = s_+^{(0)} - s_-^{(0)}$$

$$\nabla \cdot \mathbf{S}^{(0)} = \boxed{\frac{B_0}{\tilde{q}\boxed{\lambda}} \frac{\partial}{\partial \psi} \left( \frac{\tilde{q}\boxed{\lambda}}{B_0} \|\nabla \psi\| S_\psi^{(0)} \right)}$$
$$\boxed{+ \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 S_p^{(0)} \right) - \frac{1}{\boxed{\lambda} p} \frac{\partial}{\partial \xi_0} \left( \boxed{\lambda} \sqrt{1 - \xi_0^2} S_\xi^{(0)} \right)}$$

configuration space

momentum space

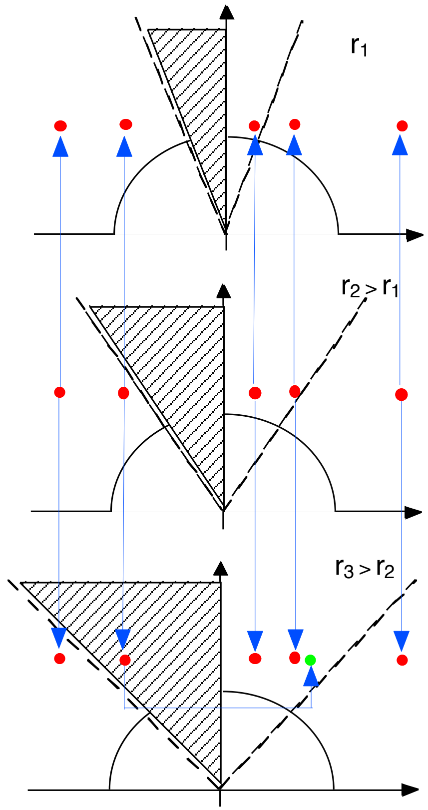
*orbit effect*

$$\mathbf{S}^{(0)} = -\mathbb{D}^{(0)} \cdot \nabla f^{(0)} + \mathbf{F}^{(0)} f^{(0)}$$

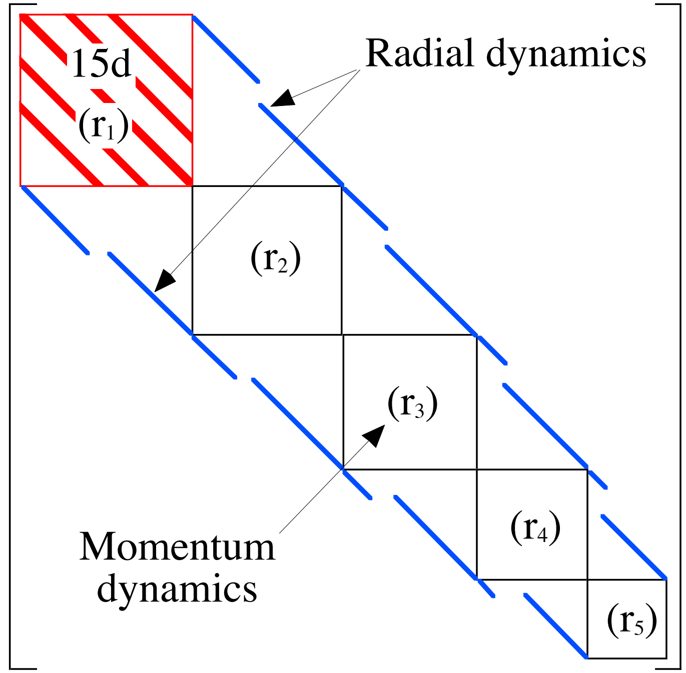
Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)

# Implicit numerical formulation of the Fokker-Planck equation

Transport induced-trapping




$$\boxed{\left(\frac{\widehat{\mathbb{A}}}{\Delta t} + \widehat{\mathbb{B}}\right)} X^{(k+1)} = \left(\frac{\widehat{\mathbb{A}}}{\Delta t}\right) X^{(k)} + \widehat{C}\left(X_M, X^{(k)}\right) + \widehat{S}_R^{(k)}$$




*~ $10^{+6}$x$10^{+6}$ entries, highly sparse matrix*

Cross-derivatives consistent with boundary conditions are *critically important* to keep the numerical solution conservative (density)

Best performances with **mul**tifrontal **m**assively **p**arallel sparse direct **s**olver (MUMPS)

Y. Peysson and J. Decker, *Fusion Science Tech.* 22, 65 (2014)

P. R. Amestoy et al., *Parallel Computing*,, 32, 136 (2006)

# Beyond standard finite-difference Fokker-Planck calculations

The challenge is to describe numerically more complex kinetic problems, *keeping the advantages of smooth distributions (finite difference,…)* with high computational performances for integrated modeling (parallel processing).

1) ***finite-banana width effects → bootstrap current, Ware pinch, rf wave induced transport,… for non-Maxwellian distributions, 3-D problem***
2) high collisionality regime (SOL) → **4-D/5-D gyrokinetic problem**
3) 3-D magnetic configurations (MHD, turbulence, magnetic ripple, other magnetic configuration) → **5-D problem (guiding-center)**

Finite-banana width effects have been investigated numerically for non-Maxwellian distributions in the thin banana width limit → **small drift approximation**.

$$\mathbf{v}_{cg} \simeq p_{||}\hat{\mathbf{b}}/\gamma + \mathbf{v}_D \qquad \|\mathbf{v}_D\| \,/\, \|\mathbf{v}_{cg}\| \ll 1$$

N. J. Fisch and J. M. Rax, *Nucl. Fusion*, 32, 549 (1992)

E. Nilsson, et al., *J. Plasma Phys* 1 (2015)

# Drift kinetic Fokker-Planck equations and bootstrap current calculations with rf waves

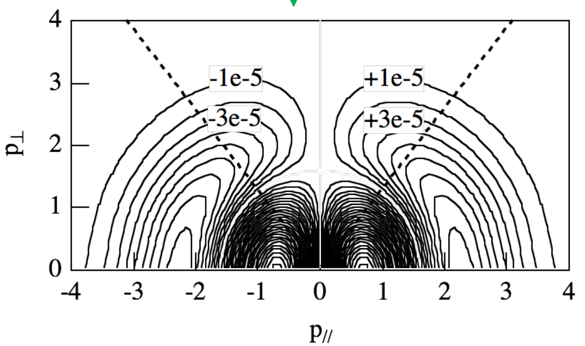
$$\partial\{f_0\}/\partial t = \{C(f_0)\} + \{Q(f_0)\} + \{E(f_0)\}$$

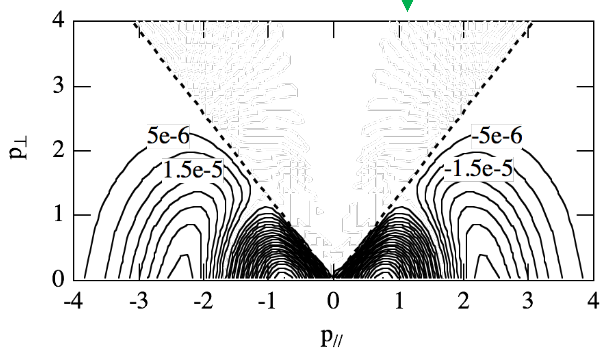
$$\tilde{f} = \frac{v_\parallel}{\Omega_e} I(\psi) \boxed{\frac{\partial f_0}{\partial \psi}}$$

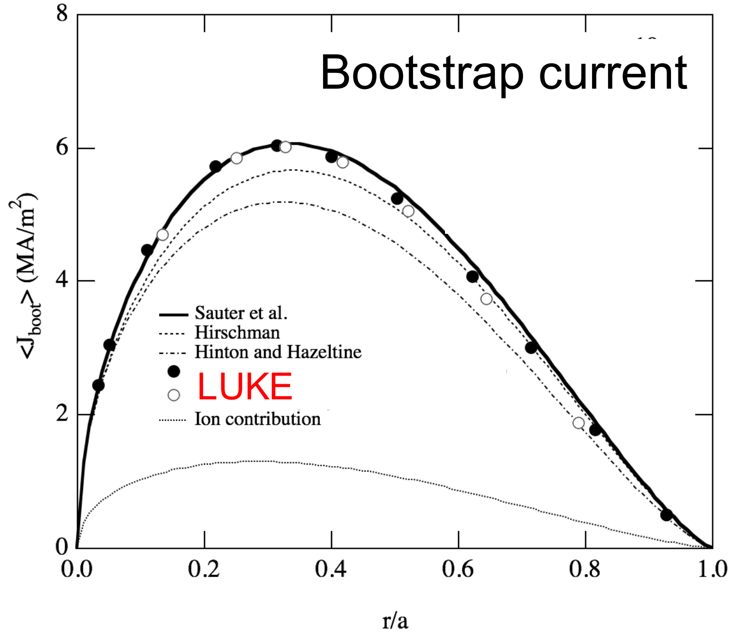
*4 Fokker-Planck equations to be solved for 1 effective radial position*

$$\{\mathcal{C}(g)\} + \{\mathcal{Q}(g)\} + \{\mathcal{E}(g)\} = -\left\{\mathcal{C}\left(\tilde{f}\right)\right\} - \left\{\mathcal{Q}\left(\tilde{f}\right)\right\} - \left\{\mathcal{E}\left(\tilde{f}\right)\right\}$$

$$f = f_0 + f_1 = f_0 + \boxed{\tilde{f}} + \boxed{g}$$



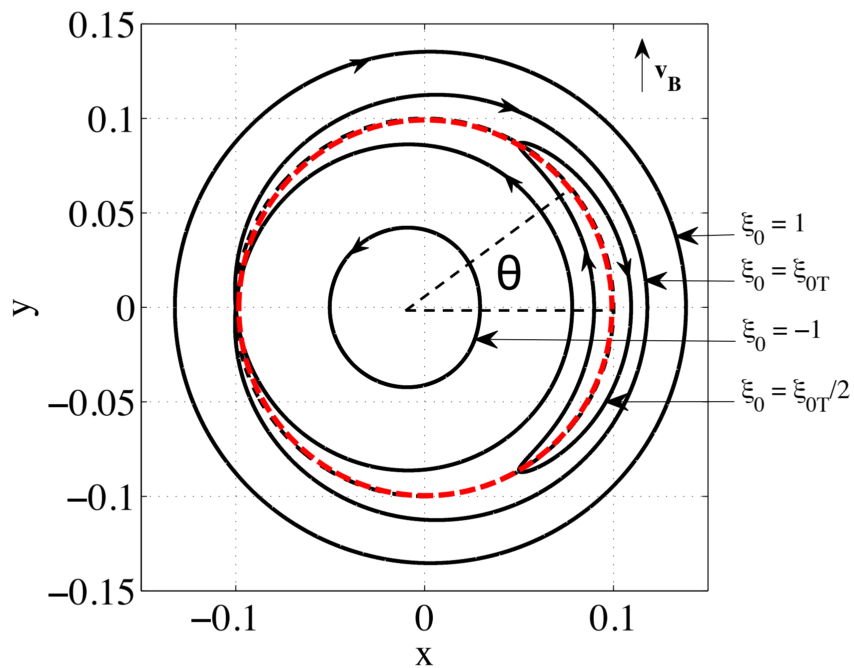




Y. Peysson et al., AIP Conf. Proc. **787**, 269 (2005)

*Used for rf current drive in ITB's*

# Finite banana width effects from an exact guiding-center Hamiltonian




Derive an exact guiding-center Hamiltonian and Poisson bracket (Lie transform)

$$\frac{\partial f_{gc}}{\partial t} + \{f_{gc}, H_{gc}\}_{gc} = C_{gc}\{f_{gc}, H_{gc}\}$$

$$C_{gc}[f] = -\frac{1}{\mathcal{J}_{gc}}\frac{\partial}{\partial Z^\alpha}\left[\mathcal{J}_{gc}\left(K^\alpha_{gc} f - D^{\alpha\beta}_{gc}\frac{\partial f}{\partial Z^\beta}\right)\right]$$

$$\langle\ldots\rangle_{\mathcal{O}} \equiv \frac{1}{\tau_{\mathcal{O}}}\oint_{\mathcal{O}}\frac{d\theta}{\dot{\theta}}$$ orbit averaging (num.)

$$C^{(0)}_{gc}\left[f^{(0)}\right] = -\frac{1}{\mathcal{J}_{\mathcal{O}}}\frac{\partial}{\partial I^a}\underbrace{\left[\mathcal{J}_{\mathcal{O}}\left(K^{a(0)}_{gc} f^{(0)} - D^{ab(0)}_{gc}\frac{\partial f^{(0)}}{\partial I^b}\right)\right]}_{\text{divergence form}} + \boxed{M^{(0)}_{gc} f^{(0)}}$$

Brizard, et al.., *Phys. Plasmas*, 16, 102304 (2009)

# Convection vector and diffusion tensor in the thin orbit approximation

$$
\begin{cases}
\mathcal{K}_{gc}^{\overline{\psi}} = \epsilon_\psi \delta\psi \left[\nu_l + \frac{1}{\xi}\nu_t\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{K}_{gc}^{p} = -\left[1 - \epsilon\lambda_{gc}\frac{1-\xi^2}{2\xi}\frac{\partial}{\partial\xi}\right]\nu_l p + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{K}_{gc}^{\xi_0} = -\frac{1-\xi_0^2}{2\xi_0}\left[\left(\epsilon_\psi\overline{\delta\psi} + \epsilon\lambda_{gc}\right)\nu_l \right.\\
\qquad \left. + \frac{2\xi}{1-\xi^2}\left(1 - \epsilon\lambda_{gc} - \epsilon\lambda_{gc}\frac{1-\xi^2}{2\xi}\frac{\partial}{\partial\xi} + \epsilon_\psi\overline{\delta\psi}\frac{1-\xi^2}{2\xi^2}\right)\nu_t\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right)
\end{cases}
$$

Thin orbit approximation: $\epsilon_\psi \equiv \left(\psi - \overline{\psi}\right)/\overline{\psi} \ll 1$

Magnetic non-uniformity: $\epsilon \equiv \rho/L_B \ll 1$

$$
\begin{cases}
\mathcal{D}_{gc}^{\overline{\psi}\overline{\psi}} = \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{D}_{gc}^{pp} = \left[1 - \epsilon\lambda_{gc}\frac{1-\xi^2}{2\xi}\frac{\partial}{\partial\xi}\right] D_l + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{D}_{gc}^{\xi_0\xi_0} = \frac{1-\xi_0^2}{p^2}\frac{\xi^2}{\Psi\xi_0^2}\left[\left(1 - \epsilon\lambda_{gc} - \epsilon\lambda_{gc}\frac{1-\xi^2}{2\xi}\frac{\partial}{\partial\xi} + \epsilon_\psi\overline{\delta\psi}\frac{1-\xi^2}{\xi^2}\right) D_t \right.\\
\qquad \left. + \frac{1}{\xi}\left(\epsilon\lambda_{gc} + \epsilon_\psi\overline{\delta\psi}\right) D_\times\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{D}_{gc}^{p\overline{\psi}} = -\epsilon_\psi\frac{\delta\psi}{p}\left[D_l + \frac{1}{\xi}D_\times\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{D}_{gc}^{p\xi_0} = \frac{1-\xi_0^2}{2p\xi_0}\left[\left(\epsilon_\psi\overline{\delta\psi} + \epsilon\lambda_{gc}\right) D_l \right.\\
\qquad \left. + \frac{2\xi}{1-\xi^2}\left(1 - \epsilon\lambda_{gc} - \epsilon\lambda_{gc}\frac{1-\xi^2}{2\xi}\frac{\partial}{\partial\xi} + \epsilon_\psi\overline{\delta\psi}\frac{1-\xi^2}{2\xi^2}\right) D_\times\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right) \\
\mathcal{D}_{gc}^{\overline{\psi}\xi_0} = -\epsilon_\psi\delta\psi\frac{1-\xi_0^2}{p^2\xi_0}\left[D_t + \frac{\xi}{1-\xi^2}D_\times\right] + \mathcal{O}\left(\epsilon^2, \epsilon\epsilon_\psi, \epsilon_\psi^2\right)
\end{cases}
$$

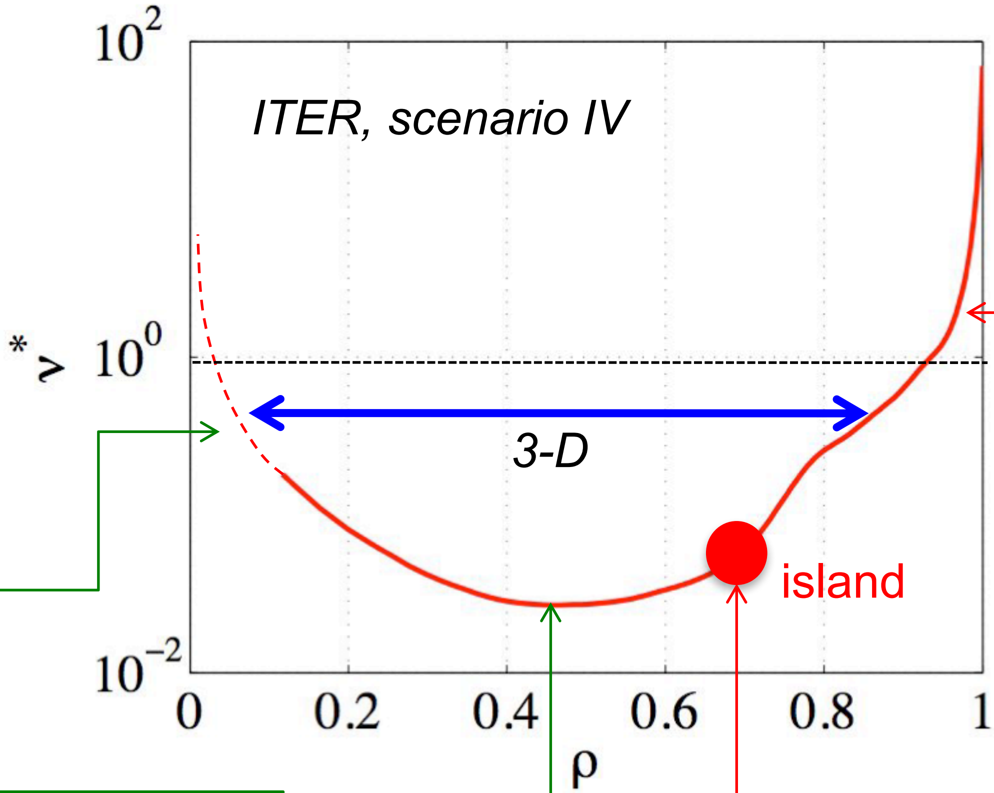
Classical Maxwellian bootstrap current recovered analytically (Lorentz collision operator)

**LUKE 2**: numerical algorithm of LUKE preserved, but more off-diagonals elements, Jacobian modified, and the physical meaning of ξ is no more a cosine of the pitch-angle

J. Decker, Y. Peysson, et al., *Phys. Plasmas* 17, 112513 (2010)

# Solved and unsolved problems for Fokker-Planck calculations in tokamaks

Particle tracking codes (anomalous orbits)




4-D/5-D gyrokinetic FEM code for SOL

$t > t_{toroidal}$

3-D finite-difference codes (zero-banana width). *Finite-orbit width effects soon implemented for LUKE code (regular orbits)*

5-D guiding-center calculations (finite-difference method)

W A Hornsby, et al., *Plasma Phys. Contr. Fusion* 52, 075011 (2007)